Chapter 2

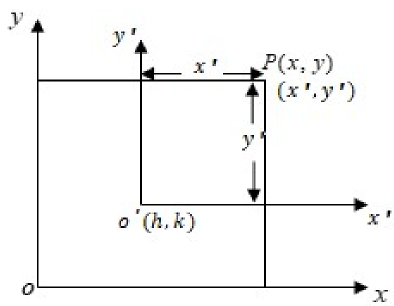
Transformation of Axes

2.1 Transformation of Coordinates

It sometimes happens that the choice of axes at the beginning of the solution of a problem does not lead to the simplest form of the equation. By a proper transformation of axes an equation may be simplified. This may be accomplished in two steps, one called translation of axes, the other rotation of axes.

2.1.1 Translation of axes

Let ox and oy be the original axes and let $o'x'$ and $o'y'$ be the new axes, parallel respectively to the old ones. Also, let $o'(h,k)$ referred to the origin of new axes.



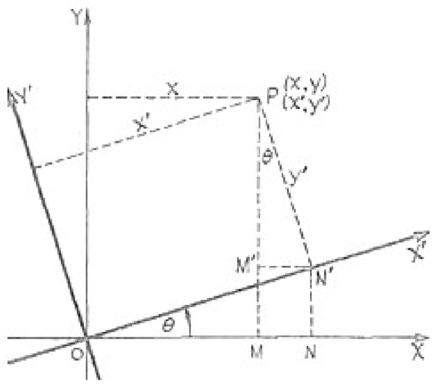
Let P be any point in the plane, and let its coordinates referred to the old axes be (x,y) and referred to the new axes be $(x'y')$. To determine x and y in terms of x',y',h and

$$x = h + x' \quad y = k + y'$$

these equations represent the equations of translation and hence, the coefficients of the first degree terms are vanish.

2.1.2 Rotation of axes

Let ox and oy be the original axes and ox' and oy' the new axes. 0 is the origin for each set of axes. Let the angle x' ox through which the axes have been rotated be represented by θ. Let P be any point in the plane, and let its coordinates referred to the old axes be (x, y) and referred to the new axes be (x', y'), To determine x and y in terms of x', y' and θ :



$$\begin{aligned} x &= OM \\ &= ON - MN \\ &= x' \cos\theta - y' \sin\theta \end{aligned}$$

and

$$\begin{aligned} y &= MP \\ &= MM' + M'P \\ &= NN' + M'P \\ &= x' \sin\theta + y' \cos\theta \end{aligned}$$

Hence the formulas for the rotation of the axes through an angle θ are

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta$$

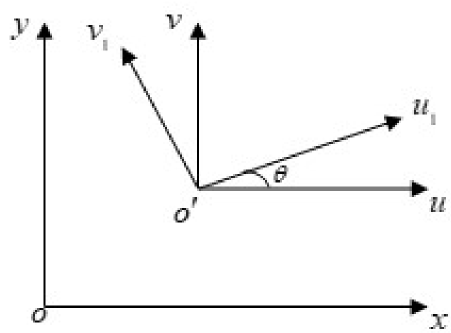
Example 2.1.1. *Determine the equation of the curve* $2x^2 + 3y^2 - 8x + 6y = 7$ *when the origin is translated to the point* $(2, -1)$.

Solution:

Substituting $x = x' + 2, y = y' - 1$, *in the curve equation, we obtain*

$$2\left(x' + 2\right)^2 + 3\left(y' - 1\right)^2 - 8\left(x' + 2\right) + 6\left(y' - 1\right) = 7$$

Expanding and simplifying, the equation of the curve referred to the new axes is



$2x^2 + 3y^2 = 18$

This is the standard equation of the ellipse with its center at the new origin and its major axis on the x - axis, with semi axes $a = 3, b = \sqrt{6}$

Example 2.1.2. *Determine a translation of axes that will transform the equation* $3x^2 - 4y^2 + 6x + 24y = 135$ *into one in which the coefficients of the first degree terms are zero.*

Solution:

Substitute for x and y the values $x' + h$ *and* $y' + k$ *respectively and collect the coefficients of the various powers of* x' *and* y'

$$3\left(x' + h\right)^2 = 4\left(y' + k\right)^2 + 6\left(x' + h\right) + 24\left(y' + k\right) = 135$$
$$3x'^2 = 4y'^2 + (6h + 6)x' - (8k - 24)y' + 3h^2 - 4k^2 + 6h + 24k = 135$$

from $6h+6=0$ *and* $8k-24=0$ *we obtain* $h=-1$ *and* $k=3$*, and the equation becomes*

$$3x^2-4y'^2=102$$

This is the standard form for the hyperbola with its center at the origin, transverse axis on the x -axis, and semi-transverse axis = $\sqrt{34}$

Another method. *The following method is often used to eliminate first degree terms. By completing the square,* $3x^2-4y^2+6x+24y=135$ *becomes*

$$\begin{aligned} 3\left(x^2+2x+1\right)-4\left(y^2-6y+9\right)&=102 \\ 3(x+1)^2-4(y-3)^2&=102 \end{aligned}$$

For $x+I$ *substitute x ′, and for* $y-3$ *substitute* y'*. Then the equation becomes*

$$3x'^2-4y'^2=102$$

Example 2.1.3. *Delete the coefficient of the first degree terms in the equation*

$$4x^2+y^2-16x+6y+9=0$$

Solution: *Let the new coordinates are* u,v *and the new origin is* (α,β)

$$x=u+\alpha \quad , \quad y=v+\beta$$

Then

$$\begin{aligned} 4(u+\alpha)^2+(v+\beta)^2-16(u+\alpha)+6(v+\beta)+9&=0 \\ 4u^2+v^2+(8\alpha-16)u+(2\beta+6)v+4\alpha^2+\beta^2&=0 \\ -16\alpha+6\beta+9&=0 \end{aligned}$$

To delete the coefficients of the first degree terms, we put

$$8\alpha-16=0 \Rightarrow \alpha=2 \quad , \quad 2\beta+6=0 \Rightarrow \therefore \beta=-3$$

Then, the equation becomes

$$\begin{aligned} 4u^2+v^2+(16+9-32-18+9)&=0 \\ 4u^2+v^2&=16 \end{aligned}$$

Example 2.1.4. *Find the equation of the curve $x^2 + y^2 + 2gx + 2fy + c = 0$, if the origin translated into $o'(-g, -f)$ and the new axes parallel to the old ones.*

Solution:

Let the new coordinates are u, v then $x = u - g, y = v - f$

$$(u-g)^2 + (v-f)^2 + 2g(u-g) + 2f(v-f) + c = 0$$
$$u^2 + v^2 = g^2 + f^2 - c$$

Example 2.1.5. *If the axes rotate by angle $\pi/4$, find the next equation in the new coordinates.*

$$5x^2 + 2xy + 5y^2 = 2$$

Solution:

Let the new coordinates are u, v then

$$x = u\cos\frac{\pi}{4} - v\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(u-v)$$
$$y = u\sin\frac{\pi}{4} + v\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}(u+v)$$

Hence

$$\frac{5}{2}(u-v)^2 + (u-v)(u+v) + \frac{5}{2}(u+v)^2 = 2$$
$$3u^2 + 2v^2 = 1$$

We note that, the axes rotation deletes the term xv

Example 2.1.6. *Prove that, the term xy in equation*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

can be deleted if axes rotate by angle $\tan 2\alpha = 2h/(a-b)$, hence delete the term xy from equation

$$14x^2 - 4xy + 11y^2 = 5$$

Solution:

if the axes rotate by angle α, then

$$x = u\cos\alpha - v\sin\alpha$$
$$y = u\sin\alpha + v\cos\alpha$$

Hence

$$a(u\cos\alpha - v\sin\alpha)^2 + 2h(u\cos\alpha - v\sin\alpha)(u\sin\alpha + v\cos\alpha)$$
$$+ b(u\sin\alpha + v\cos\alpha)^2 + 2g(u\cos\alpha - v\sin\alpha) + 2f(u\sin\alpha + v\cos\alpha) + c = 0$$

The coefficient of the term uv is

$$-2a\sin\alpha\cos\alpha + 2h\left(\cos^2\alpha - \sin^2\alpha\right) + 2b\sin\alpha\cos\alpha$$

This term vanishes if

$$2h\cos 2\alpha = (a-b)\sin 2\alpha$$
$$\tan 2\alpha = 2h/(a-b)$$

If $a = 14, b = 11$ *and* $h = -2$

$$\tan 2\alpha = 2h/(a-b) = -4/(14-11) = -4/3$$

$$2\tan^2\alpha - 3\tan\alpha - 2 = 0$$
$$(\tan\alpha - 2)(\tan\alpha + 1) = 0$$

Hence $\tan\alpha = 2 \quad \tan\alpha = -1/2$
If we take tan $\alpha = 2, \cos\alpha = 1/\sqrt{5}, \sin\alpha = 2/\sqrt{5}$

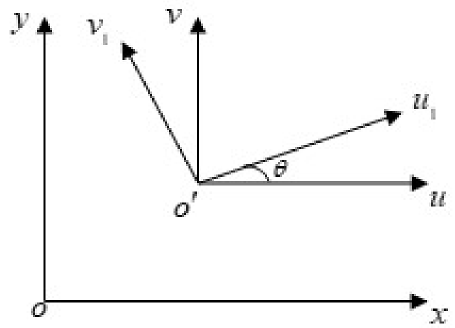
$$x = \frac{1}{\sqrt{5}}(u - 2v) \quad , \quad y = \frac{1}{\sqrt{5}}(2u + v)$$

$$\frac{14}{5}(u-2v)^2 - \frac{4}{5}(u-2v)(2u+v) + \frac{11}{5}(2u+v)^2 = 5$$
$$2u^2 + 3v^2 = 1$$

If we take tan $\alpha = -1/2$, *we will get the same result because this means that two axes are replaced.*

2.1.3 Rotation and translation of axes

Let the origin is translated into the point $o'(\alpha, \beta)$, and then the axes are rotate by angle θ Then the equations of the rotation and translation of axes are



$$x = u + \alpha = u_1 \cos\theta - v_1 \sin\theta + \alpha$$
$$y = v + \beta = u_1 \sin\theta + v_1 \cos\theta + \beta$$

Example 2.1.7. *If the origin is translated to the point* $(-1, 2)$ *and the axes are rotate by angle* 45°*, find the new coordinates for the point* $(1, 3)$ *and the new equation for the curve*

$$4x^2 + y^2 + 8x - 4y + 7 = 0$$

Solution:
To find the new coordinates for the point , we put in the previous relations

$$x = u_1 \cos 45^\circ - v_1 \sin 45^\circ - 1 = \frac{1}{\sqrt{2}}(u_1 - v_1) - 1$$
$$y = u_1 \sin 45^\circ + v_1 \cos 45^\circ + 2 = \frac{1}{\sqrt{2}}(u_1 + v_1) + 2$$

Hence

$$1 = \frac{1}{\sqrt{2}}(u_1 - v_1) - 1 \quad \Rightarrow \quad u_1 - v_1 = 2\sqrt{2}$$
$$3 = \frac{1}{\sqrt{2}}(u_1 + v_1) + 2 \quad \Rightarrow \quad u_1 + v_1 = \sqrt{2}$$

By solving these two equations, we get

$$u_1 = \frac{3\sqrt{2}}{2} \quad , \quad v_1 = -\frac{\sqrt{2}}{2}$$

To find the new equation for the curve

$$4x^2 + y^2 + 8x - 4y + 7 = 0$$

we first consider the translation of origin, $x = u - 1 \quad , \quad y = v + 2$, *then*

$$\begin{aligned} 4(u-1)^2 + (v+2)^2 + 8(u-1) - 4(v+2) + 7 = 0 \\ 4u^2 + v^2 = 1 \end{aligned}$$

Second, we consider the rotation of axes

$$\begin{aligned} u = u_1 \cos 45^\circ - v_1 \sin 45^\circ - 1 &= \frac{1}{\sqrt{2}}(u_1 - v_1) \\ v = u_1 \sin 45^\circ + v_1 \cos 45^\circ + 2 &= \frac{1}{\sqrt{2}}(u_1 + v_1) \end{aligned}$$

Hence

$$\begin{aligned} 2(u_1 - v_1)^2 + \frac{1}{2}(u_1 + v_1)^2 = 1 \\ 5u_1^2 - 6u_1v_1 + 5v_1^2 = 2 \end{aligned}$$

2.2 Exercises

1. Determine the equation of the parabola $x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ when the axes have been rotated 45°.

2. Determine the angle through which the axes must be rotated to remove the xy term in the equation $7x - 6\sqrt{3}xy + 13y^2 = 16$.

3. By translation and rotation of the axes reduce the equation $5x^2 + 6xy + 5y^2 - 4x + 4y - 4 = 0$ to its simplest form. Sketch the curve showing all three sets of coordinate axes.

4. Remove the first degree terms in

   (a) $3x^2 + 4y^2 + 4y + 13 = 0$
   (b) $x^2 + 2xy - 2y^2 + 4x + 2y + 6 = 0$
   (c) $9x^2 - 4xy + 6y^2 - 10x - 7 = 0$
   (d) $3x^2 - xy + 3y^2 - 2x - 2y = 0$

5. Find the point of translation which transform the equation $x^2 + 2y^2 + 3x - 4y + 7 = 0$ into $x^2 + 2y^2 + c = 0$ and find value of c

6. Determine the angle through which the axes must be rotated to remove the xy term in the equation $x^2 - xy + 2y^2 = 6$

7. Determine the equation $x^2 + 2xy + y^2 = 1$ when the axes have been rotated 90°

8. If the origin is translated to the point $(1, -3)$ and the axes are rotate by angle $\tan^{-1} 3/4$, find the new coordinates for the point $(2, -2)$ and the new equation for the curve

$$36x^2 + 24xy + 29y^2 + 150y + 45 = 0$$

9. Determine the equation of the straight line $x = 2y$ and the curve

$$x^2 - 4xy + y^2 = 1$$

when the axes have been rotated 90°