On Some Dynamical Properties of Discontinuous Dynamical Systems

A. M. A. El-Sayed and M. E. Nasr

1 Faculty of Science, Alexandria University, Alexandria 21511, Egypt
2 Faculty of Science, Benha University, Benha 13518, Egypt

Corresponding author: Email: amasayed5@yahoo.com

Abstract: In this work we are concerned with the discontinuous dynamical system representing the problem of the logistic retarded functional equation with delay $r > 0$. The existence and uniqueness of the solution will be proved. The local stability at the equilibrium points will be studied. The bifurcation analysis and chaos will be discussed.

Keywords: Logistic functional equation, existence, uniqueness, equilibrium points, local stability, bifurcation, chaos.

1 Introduction

Consider the problem of retarded functional equation,

$$\begin{align*}
  x(t) &= f(x(t-r)), \quad t \in (0, T] \\
  x(t) &= x_0, \quad t \leq 0.
\end{align*}$$

Let $t \in (0, r]$ , then $t-r \in (-r, 0]$ and the solution of (1) is given by

$$x(t) = x_r(t) = f(x_0), \quad t \in (0, r].$$

For $t \in (r, 2r]$, we find that $t-r \in (0, r]$ and the solution of (1) is given by

$$x(t) = x_{2r}(t) = f(x_r(t)) = f^2(x_0), \quad t \in (r, 2r].$$

Repeating the process we can deduce that the solution of the problem (1) is given by

$$x(t) = x_{nr}(t) = f^n(x_0), \quad t \in ((n-1)r, nr],$$

which is continuous on each subinterval $((k-1)r, kr), \quad k = 1, 2, ..., n$, but

$$\lim_{t \to kr} x_{(k+1)r}(t) = f^{k+1}(x_0) \neq x_{kr}(t),$$

which implies that the solution of the problem (1) is discontinuous (sectionally continuous) on $(0, T]$.

So, we can give the following definition,

**Definition 1** The discontinuous dynamical system is a problem of retarded functional equation,

$$\begin{align*}
  x(t) &= f(t, x(t-r_1), x(t-r_2), ..., x(t-r_n)), \quad t \in (0, T] \\
  x(t) &= x_0, \quad t \leq 0.
\end{align*}$$

2 Existence and Uniqueness

Let $L^1 = L^1[0, T]$ , $T < \infty$ be the class of Lebesgue integrable functions on $[0, T]$ with norm

$$\|f\| = \int_0^T |f(t)| dt, \quad f \in L^1.$$
Let $D = \{ x \in \mathbb{R} : 0 \leq x(t) \leq 1, t \in (0,T) \}$ and $x(t) = x_0, t \leq 0$.

**Definition 3** By a solution of the problem (4)–(5) we mean a function $x \in L^1$ satisfying the problem (4)–(5).

**Theorem 1** The problem (4)–(5) has a unique solution $x \in L^1$.

**Proof.** Define, on $D$, the operator $F : L^1 \rightarrow L^1$ by

$$F(x)(t) = \rho x(t-r)[1-x(t-r)].$$

The operator $F$ makes sense, indeed for $x \in D$ we have

$$\| F(x)(t) \| \leq \rho \| x(t-r) \|,$$

and

$$\| F(x)(t) \| \leq r \rho x_0 + \rho \| x \|.$$

Now for $x, y \in D$, we can obtain

$$|F(x)(t) - F(y)(t)| \leq \rho |x(t-r)(1-x(t-r)) - y(t-r)(1-y(t-r))| \leq \rho |x(t-r) - y(t-r)|,$$

which implies that

$$|F(x)(t) - F(y)(t)| \leq \rho \int_0^t |x(t-r) - y(t-r)| dt = \rho \int_0^t [F(x)(t) - F(y)(t)] dt \leq \rho \int_0^t |x(t-r) - y(t-r)| dt \leq \rho \| x - y \|.$$  

If $\rho < 1$, we deduce that

$$\| F(x) - F(y) \| \leq \| x - y \|$$

and then the problem (4)–(5) has, on $D$, a unique solution $x \in L^1$. □

**3 Continuous dependence on initial conditions**

**Theorem 2** If $\rho < 1$. Then the solution of the discontinuous dynamical system represents the problem of the logistic retarded functional equation with delay (4)–(5) is continuously dependent on the initial data in the sense that,

$$|x(t) - x_0(t)| \leq \delta \Rightarrow \| x - x_0 \| \leq \varepsilon$$

where $x_0$ is the solution of the problem,

$$x(t) = \rho x(t-r)[1-x(t-r)], \quad t \in (0,T],$$

$x(t) = x_0, t \leq 0$.  

(7)

**Proof.** Let $x(t)$ and $x_0(t)$ be the solution of the two problems (4)–(5) and (4)–(7) respectively, then,

$$|x(t) - x_0(t)| \leq \rho |x(t-r) - x_0(t-r)|,$$

which implies that

$$x(t) - x_0(t) \leq \rho \int_0^t |x(t-r) - x_0(t-r)| dt = \rho \int_0^t |x(t-r) - x_0(t-r)| dt \leq \rho \left( \int_0^t (t-r) dt + \int_0^t (x(t-r) - x_0(t-r)) dt \right) \leq \rho \left( \int_0^t (t-r) dt + \| x - x_0 \| \right) = \rho \left( \frac{1}{2} t^2 - rt + \| x - x_0 \| \right).$$

and

$$\| x - x_0 \| \leq \frac{\rho r}{1-\rho} |x_0 - x_0^*|,$$

which proves that

$$|x(t) - x_0(t)| \leq \delta \Rightarrow \| x - x_0 \| \leq \varepsilon = \frac{\rho r}{1-\rho} \delta,$$

and the theorem is proved. □

**4 Equilibrium Points and their asymptotic stability**

The equilibrium points of (4) are the solution of the equation

$$\rho x_{eq}(1-x_{eq}) = x_{eq},$$

which are

$$x_{eq1} = 0,$$

$$x_{eq2} = 1 - \frac{1}{\rho}.$$

The equilibrium point of (4) is locally asymptotically sable if all the roots $\lambda$ of the equation,

$$\lambda' = \rho(1-2x_{eq}),$$

satisfy $|\lambda| < 1$ (see [8]).

Then the equilibrium point $x_{eq} = 0$ is locally asymptotically sable if $\rho < 1$, while the second equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$ is locally asymptotically sable if all the roots $\lambda$ of the equation,

$$\lambda' = \rho(1-2(1-\frac{1}{\rho}))) = 2 - \rho,$$

satisfy $|\lambda| < 1$.

The equilibrium point $x_{eq} = 0$ is locally asymptotically sable if $\rho < 1$, which is the same as in the discrete case (6). Also, when $r = 1$, we deduce that the equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$, $\rho > 1$ is locally asymptotically sable if $1 < \rho < 3$, which is the same as in the discrete case (6).

In studying (4)–(5) it may be useful to study the difference equation (6).
5 Bifurcation and Chaos

In this section, some numerical simulations results are presented to show that dynamics behaviors of the discontinuous dynamical system (4)–(5) change for different values of \( r \) and \( T \). To do this, we will use the bifurcation diagrams as follow:

**Example 1**
1. we take \( r = 1 \) and \( t \in [0, 50] \), in this case, we get the same behavior as in the discrete case (Figure 1).
2. we take \( r = 2 \) and \( t \in [0, 50] \) (Figure 2).
3. we take \( r = 1.75 \) and \( t \in [0, 50] \) (Figure 3).

**Example 2**
1. we take \( r = 0.1 \) and \( t \in [0, 5] \) (Figure 4).
2. we take \( r = 0.2 \) and \( t \in [0, 5] \) (Figure 5).
3. we take \( r = 0.3 \) and \( t \in [0, 5] \) (Figure 6).
Figure 6: Bifurcation diagram of map (4)-(5) with respect to $\rho$, $r = 0.3$ and $t \in [0,5]$.

From Figures (1–6) we deduce that the change of $r$ and $T$ effect of stability of the Logistic equation model, occurs of a bifurcation point, parameter sets for which a periodic behavior occur and parameter sets for which a chaotic behavior occur.

6 Conclusions

The discrete dynamical system of the Logistic equation model describes the dynamical properties for the case $r = 1$ and the time is discrete $t = 1,2,\ldots$.

On the other hand, the discontinuous dynamical system of the Logistic equation model describes the dynamical properties for different values of the delayed parameter $r \in R^+$ and the time $t \in [0, T]$ is continuous.

Figures (1),(4) agrees with the results of the asymptotic stability, this confirm that our numerics are correct. Also from figures (1),(4) and (2),(5), it locks like that there is a scale that gives identical chaos behavior.

This shows the richness of the models of discontinuous dynamical systems.

References


