The AIUB Journal of Science and Engineering (AJSE)

August 2008

Papers

Semiconductor Laser Dynamics under Optical Feedback: I. Type of Transition to Chaos in FP Lasers
Moustafa Ahmed, Minoru Yamada and Salah Abdulrahmann

Effect of Refractive Indices of DBR Layers on the Reflectivity of a VCSEL
Rinku Basak and Saiful Islam

Semiconductor Laser Dynamics under Optical Feedback: II. Influence of the Linewidth Enhancement Factor in Fiber-Grating Lasers
Salah Abdulrahmann, Minoru Yamada and Moustafa Ahmed

Design and Implementation of a CPLD-Based Field-Oriented Control IC for the Speed Control of 3-Phase Induction Motors
C.P. OOI, W.P. Hew, N.A. Rahim and LC Kuan

Effects of Geometrical Structures on THz range Ultra-Wideband Characteristics of On-Chip Self-Complementary Antennas Integrated with Semiconductor Mesa Structures
Hirote Tomioha, Michihiko Sahara and Tsugunori Okamura

Design and Simulation of an Improved PI Speed Control of Indirect Field-Oriented Induction Motor
Mohammad Abdul Mannan, Toshiaki Murata and Junji Tamura

Effect of Processing Gain on Broadband CDMA Systems
Md. Anwar Hossain, Poornpat Saengudomlert and Tarique Mohammed

Study of the Effects of Injection Current on the Turn-on Delay and Modulation Speed of a VCSEL
Ashim Kumar Saha and Saiful Islam

A Novel Inverter Switched Single Phase AC Voltage Rectifier
Amzad Ali Sarkar

Pattern of Change: Wall [Reinvented] in 20th Century Architecture
Mohammad Arefeen Ibrahim, M. Saleh Uddin

Infra-red Study of Carbons Obtained from Aromatic Organic Compounds with AlCl3 as Additives
M. A. Rashid, Tafazzal Hossain and M. A. Asgar

Sensitive Tint Polarized Light Microscopy–A Novel Technique for Identifying Graphitic Carbon
Jiban Podder, Syeda Tasmin Jahan and Tafazzal Hossain

An Approximate Solution for Fredholm Integral Equation of the Second Kind in the Space
$\int_{0}^{2\pi} [0, 2\pi] \text{with Weight Function } f(x)$
S. A. Abou Auf and M. E. Nasr

Unsteady MHD Forced Convective Laminar Flow for a Vertical Porous Plate in Presence of Viscous Dissipation with Variable Viscosity and Thermal Conductivity
Kh. Abdul Maleque

Comparison of Question Answering Techniques
Mohammad Farhad Ahmed, Mashhour Rahman and Md. Musfiq Rahman

Pages
1
7
13
21
25
31
39
43
49
53
61
65
69
75
81
An Approximate Solution for Fredholm Integral Equation of the Second Kind in the Space $L_p^2[0, 2\pi]$ with Weight Function $p(x)$

S. A. Abou Auf and M. E. Nasr

Abstract— In the present paper we study the approximate solution for Fredholm integral equation of the second kind in the space $L_p^2([0, 2\pi])$ with weight function $p(x)$ and bounded almost everywhere on $[0, 2\pi]$. The technique of this study is based on linear polynomial operators $U_n[q(x); x]$ which generate good approximation to the function $q(x)$ in the space $L_p^2([0, 2\pi])$, where the given equation is replaced by Fredholm integral equation with degenerate kernel. The solution of the new equation is taken as an approximate solution to the original equation, and also we give estimates of the errors which arise in this connection. This approximation is discussed in details for Dirichlet, Vallée-Poussin, Fejér, Riesz and Jackson operators.

Index Terms— Solution of Fredholm integral equation of the second kind, Linear polynomial operators, space with weight function.

I. INTRODUCTION

The approximate solution of linear integral equations have been studied by many authors in the literature; see [1], [4], [8]. In this paper, we consider the following Fredholm integral equation of the second kind,

$$
\phi(x) = f(x) + \lambda \int_0^{2\pi} k(x, y) \phi(y) \, dy,
$$

(1)

in which all functions are $2\pi$-periodic with respect to $x$ and $y$, $f(x)$ is in $L_p^2[0, 2\pi]$, $\lambda$ is some complex number such that $|\lambda|$ is a regular value of the kernel $k(x, y)$, and the kernel $k(x, y)$ satisfies the following conditions ($A^n$):

i) $\|k(x, y)\| L_{L_p^2[Q]}^1 < 1,$

ii) the functions:

$$
\mathcal{A}(c) = \left[ \int_0^{2\pi} \int_0^{2\pi} p(x) p(y) [k(x, y) \phi(y)]^2 \, dy \, dx \right]^{1/2},
$$

(3)

are bounded almost everywhere by the number $M$.

$$
\text{ess sup} A(x) \leq M, \quad \text{ess sup} B(y) \leq M.
$$

Instead of equation (1), we solve the following equation

$$
\phi_n(x) = U_n(f(x)) + \lambda \int_0^{2\pi} U_n[k(\cdot, y); x] \phi_n(y) \, dy,
$$

(4)

the notation $U_n[k(\cdot, y); x]$ will mean that the operator $U_n$ acts on $k(t, y)$ as a function of $t$, and at the same time, the variable $y$ plays the role of a parameter.

Now, since the functions $U_n(f(x))$ and $U_n[k(\cdot, y); x]$ are both trigonometric polynomials of order $n$ with respect to $x$, the solution $\phi_n(x)$ of equation (4) will also be trigonometric polynomial of order $n$.

More precisely, it is shown in this paper that:

$$
\|\phi - \phi_n\| L_{L_p^2[Q]}^1 \leq (1 + \alpha_n(k)) \|\phi - U_n(\phi; x)\| L_{L_p^2[Q]}^1, \quad \alpha_n \to 0
$$

where

$$
\|\phi(x)\| L_{L_p^2[Q]}^1 = \|\phi(x)\| L_{L_p^2[Q]}^1 = \left[ \int_0^{2\pi} p(x) |\phi(x)|^2 \, dx \right]^{1/2}.
$$

II. PRELIMINARIES

The linear polynomial operators $U_n(g; x)$ which are good approximation to the function $g(x)$ in the space $L_p^2[0, 2\pi]$ have the form:

$$
U_n(g; x) = \frac{1}{\pi} \int_0^{2\pi} g(y) U_n(x - t) \, dt = \frac{1}{\pi} \int_0^{2\pi} g(x - t) U_n(t) \, dt,
$$

(5)

where

$$
U_n(x) = \frac{1}{2} + \sum_{k=1}^n \lambda_k^{(n)} \cos(kx),
$$

(6)

$\lambda_k^{(n)}$ are given as constants depending on the linear method.

We shall recall the most important of these methods which are essential for our purposes [2],[3],[5],[9].

1. Dirichlet method $D_n(f; x)$ (the method of partial sums):

This method is obtained by letting $\lambda_k^{(n)} = 1, \quad k = 1, 2, \ldots, n$, i.e., with the help of the Dirichlet kernel

$$
D_n(x) = \frac{1}{2} + \sum_{k=1}^n \cos(kx) = \sin\left(\frac{1}{4}(2n + 1)x\right)/\sin\left(\frac{x}{2}\right).
$$

2. Fejér method $F_n(f; x)$ (the method of arithmetic means):

This method is obtained by letting $\lambda_k^{(n)} = 1 - (k/n), \quad k = 1, 2, \ldots, n$, i.e., with the help of the Fejér kernel

S. A. Abou Auf is with Faculty of Science, Department of Mathematics, Benha University, Benha 13518, Egypt. Email: s_abouauf@yahoo.com

M. E. Nasr is with Faculty of Science, Department of Mathematics, Benha University, Benha 13518, Egypt. Email: moh_nasr_2000@yahoo.com
3. The method of Vallee-Poussin $V_n(f; x)$ is obtained with the help of the kernel

$$V_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx) = \frac{1}{2} \sum_{k=1}^{n} \cos(kx) = 2F_n(x) - F_n(x).$$

4. The method of Rogosinski $R_n(f; x)$ is obtained with the help of the kernel

$$R_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx) = \frac{1}{2} \sum_{k=1}^{n} \cos(kx/2) = \frac{\sin((n+1)x/2)}{\sin(x/2)} \cos(x) = \frac{\sin((n+1)x/2)}{\sin(x/2)} \cos(x).$$

5. The method of Jackson $J_n(f; x)$ is obtained with the help of the kernel

$$J_n(x) = \frac{1}{2} + \sum_{k=1}^{2n-2} \mu_k \cos(kx) = \frac{\sin((n+1)x/2)}{\sin(x/2)} \cos(x).$$

where the $\mu_k$ are numbers which we will not write out.

THEOREM 1. [3]
For any kernel $k(x, y) \in L^2[0, \pi]$, if the linear polynomial operators $U_n$ of order $n$ is defined in $L^2_{\text{pol}}$ and if the function $f(x) \in L^2_{\text{pol}}$, then

$$U_n \left[ \int_0^\pi k(x, y) f(y) dy \right] = \sum_{n=1}^N U_n \left[ k(x, y) f(y) dy \right].$$

THEOREM 2. [6]
If $A$ and $B$ are two bounded linear operators in Banach space $E$, while $A$ has an inverse and $\|B\|_E \leq 1$ then the operator $(A+B)^{-1}$ has an inverse and

$$\| (A+B)^{-1} \|_E \leq \| A^{-1} \|_E / (1 - \|B\|_E \| A^{-1} \|_E).$$

THEOREM 3. [7]
For $f(x)$ and $k(x, y)$ belongs to $L^2$, if $\|k(x, y)\|_{L^2} < 1$, then the integral equation

$$\phi(x) = f(x) + \frac{1}{\pi} \int_0^\pi (k(x, y) - k(x, y)) \phi(y) dy,$$

has a unique solution $\phi(x)$ in $L^2_{\text{pol}}$.

III. AUXILIARY DEFINITIONS AND THEOREMS

The discussion in the present section are stimulated and achieved by the following procedure:

**Definition 1.**
The mean-modulus of continuity of the kernel $k(x, y) \in L^2$ is defined by the function

$$\omega_{L^2}(k; t) = \omega_{L^2}(t).$$

**Definition 2.**
The value of the following norm:

$$\delta_n(k) = \delta(k; U_n) = \left\| U_n(k(x, y); x) - k(x, y) \right\|_{L^2} = \left[ \int_0^\pi \int_0^\pi (k(x, y) - k(x, y))^2 \right]^{1/2}.$$

It is evident that:

1. $\omega_{L^2}(0) = 0$
2. $\omega_{L^2}(t_1 + t_2) \leq \omega_{L^2}(t_1) + \omega_{L^2}(t_2)$.
3. $\omega_{L^2}(t)$ is positive and monotone increasing function.
4. $\omega_{L^2}(\eta t) \leq (1 + \eta) \omega_{L^2}(t)$, for any positive real number $\eta$.

**Theorem 4.**
For any square-summable kernel $k(x, y) \in L^2$, and for any linear polynomial operator $U_n(g; x)$, we always have the inequality:

$$\delta_n(k) \leq 2 \omega_{L^2}(k; \frac{1}{n}) \int_0^{\pi} \left| U_n(t) \right| (n |t| + 1) dt.$$

**Definition 3.**
We define the error of approximation of $k(x, y)$ as follows:

$$E_{n, \alpha}(k)_{L^2} = \left\| k(x, y) - T_{n, \alpha}(x, y) \right\|_{L^2} = \inf_{T_{n, \alpha}(x, y)} \left[ \int_0^\pi \int_0^\pi p(x)p(y)[k(x, y) - T_{n, \alpha}(x, y)]^2 dx dy \right]^{1/2}.$$

where $T_{n, \alpha}(x, y)$ denotes the trigonometric polynomial in $x$ of order $n$ of best approximation of $k(x, y)$ in the
metric $L^2_{p}$. Similarly, $r_{m \in m}(x, y)$ denotes the trigonometric polynomial in $y$ of order $m$ of best approximation of $k(x, y)$ in the metric $L^2_{p}$. The estimates of how rapidly the quantities $E_{m \in m}(k(x,y))$ and $E_{m \in m}(k(x,y))$ tend to zero as $n \to \infty$, $m \to \infty$ are given in [10], where

$$E_{m \in m}(k(x,y)) = 0 \quad \text{as} \quad n \to \infty.$$  

$$E_{m \in m}(k(x,y)) = 0 \quad \text{as} \quad m \to \infty.$$  

Now, we will mention bounds of the norm (8) for various linear polynomial operators $U_n$.

1. In the case of Jackson's method [2], [3], [9]: $U_n = J_n$

$$\delta(k ; J_n) \leq 12 \pi \alpha \pi \frac{1}{n} \quad \text{(1) }$$

from definition (3), then:

$$E_{m \in m}(k(x,y)) = \left\| k(x, y) - J_n [k(., y)] ; x \right\|_{L^2_{p}} \leq$$

$$= 12 \pi \alpha \pi \frac{1}{n} \quad \text{as} \quad n \to \infty.$$  

2. For the Vallée-Poussin's method [2], [3], [9]:

$$U_n = V_n = \frac{1}{\pi} \int_{0}^{2\pi} V_n(t) \ dt = \frac{3}{\pi} \approx 1.436$$

Considering that the method of Vallée-Poussin's $V_n$ leaves trigonometric polynomials of order $n$ invariant, then $V_n \left( T_{m \in m}(x, y) ; x \right) = T_{m \in m}(x, y)$ and

$$\delta(k; V_n) = \left\| k(x, y) - T_{m \in m}(x, y) - V_n [k(., y)] - T_{m \in m}(., y) ; x \right\|_{L^2_{p}} \leq$$

$$\leq E_{m \in m}(k(x,y)) + \frac{1}{\pi} \int_{0}^{2\pi} \left| V_n(t) \right| \ dt \leq$$

$$\leq 2.5 E_{m \in m}(k(x,y)) \leq 29.232 \pi \alpha \pi \frac{1}{n} \quad \text{(14) }$$

3. In the case of Dirichlet's method [2], [3], [9]: $U_n = D_n$ by considerations similar to those used in the case of Vallée-Poussin's method and after some calculations we obtain:

$$\delta(k; D_n) \leq (3 + \ln n) E_{m \in m}(k(x,y)) \leq 12 \pi (3 + \ln n) \alpha \pi \frac{1}{n} \quad \text{(15) }$$

4. In the case of Rogosinski's method [2], [3], [9]: $U_n = R_n$ we give two estimates. The first estimate easily follows from equation (11), we obtain:

$$\delta(k; R_n) \leq (4 \pi + 2 \pi^2 + \frac{\pi^3}{2} \ln 2 \pi) \alpha \pi \frac{1}{n} \quad \text{(16) }$$

In the second estimate, we let $n' = \sqrt{n}/2$,

and let $a_r(y), b_r(y), a_r^*(y), b_r^*(y)$ denote the corresponding coefficients of Fourier series in the variable $x$ of the functions $k(x, y)$ and $V_n(k(x, y))$.

Then, by using the Parseval's identity

$$\frac{a_r^*(y)}{2} + \sum_{k=1}^{2 \pi n} \left( a_k^*(y) + b_k^*(y) \right)^2 \leq \frac{1}{\pi} \int_{0}^{2\pi} \left| \rho(x) \right| \left( k(x, y) \right)^2 \ dx \quad \text{dx}$$

Therefore,

$$\left\| a_r^*(y) + \sum_{k=1}^{2 \pi n} \left( a_k^*(y) + b_k^*(y) \right) \right\|_{L^2_{p}} \leq \frac{1}{\sqrt{n}} \left\| k(x, y) \right\|_{L^2_{p}} \quad \text{(17) }$$

On the other hand, from (14) we have:

$$\delta(k; R_n) = \left\| k(x, y) - V_n(k(x, y)) ; x \right\|_{L^2_{p}} \leq R_n \left( V_n(., y) ; x \right) + R_n \left( V_n(., y) ; x \right) -$$

$$\leq 7.5 E_{m \in m}(k(x,y)) + \frac{3\pi}{8} n^{-3/4} \left\| k(x, y) \right\|_{L^2_{p}} \quad \text{(18) }$$

5. In the case of Fejér's method [2], [3], [9]: $U_n = F_n$ by consideration similar to those used in the case of Rogosinski's method we find:

$$\delta(k; F_n) \leq 5 E_{m \in m}(k(x,y)) + \frac{1}{\pi} n^{-1/4} \left\| k(x, y) \right\|_{L^2_{p}} \quad \text{(18) }$$

Now, it is clear that $\delta(k, U_n) \to 0$ as $n \to \infty$ for Jackson's, Vallée-Poussin's, Fejér's and Rogosinski's methods for every square integrable periodic function $k(x, y) \in L^2_{p}$. In the case of Dirichlet's method $\delta(k, D_n)$

$$\to 0 \quad \text{as} \quad n \to \infty \quad \text{if} \quad \alpha \pi \frac{1}{n} \to 0 \quad \text{(1 ln n)}.$$

**Definition 4.** The following quantities will play an important role in estimating the error of our approximation:

$$\zeta = \left( \int_{0}^{2\pi} \left( k(x, y) - T_{m \in m}(x, y) \right)^2 \ dy \right)^{1/2} \quad \text{on} \quad [0, 2\pi] \quad \text{(19) }$$

and $\gamma_m = \gamma_m(U_n ; \varphi) = \sum_{n=1}^{m} \left( 1 - \lambda_n(\varphi) \right) \left( E_{m \in m}(\varphi) ; L^2_{p} \right) \quad \text{(20) }$

where

$$E_{m \in m}(\varphi) = \inf_{T_{m \in m}} \left\| \varphi(., y) - T_{m \in m}(., y) \right\|_{L^2_{p}}$$

$T_{m}(x)$ is a trigonometric polynomial of order $n$ in $x$, $m \leq \infty$.

Let us denote by $W^r H^\beta(L^2_{p})$ ($\beta$- non negative integer,

$0 < \beta \leq 1$) the class of periodic functions of period $2\pi$ possessing the derivatives of order $r$, which satisfies Lipschitz condition of order $\beta$ [3], [10] and denote to:

$$\gamma_m(U_n ; W^r H^\beta(L^2_{p})) = \sum_{n=1}^{m} \left( 1 - \lambda_n(\varphi) \right) \sup_{f \in W^r H^\beta(L^2_{p})} \left( E_{m \in m}(f) \right) \left( L^2_{p} \right) \leq$$

$$\leq \frac{1}{4} \pi \left( 1 + \beta \right) m \sum_{n=1}^{m} \left( 1 - \lambda_n(\varphi) \right) \left( L^2_{p} \right) \quad \text{(21) }$$
We shall derive from this inequality the following condition
\[
\gamma_n(U_n, W^r H^\beta(L^2)) = \sup_{p(x) \in P_n} \int_{\mathbb{R}^d} \left| \frac{\partial^r}{\partial x^r} p(x) - U_n(x) \phi(x) \right|^2 \, dx \leq 0
\]  
(22)

It is clear that
\[
\gamma_n(D_n, W^r H^\beta(L^2)) = \gamma_n(V_n, W^r H^\beta(L^2)) = 0
\]
for all \( m \leq n \).

Then condition (22) will be satisfied for all \( r = 0, 1, 2, \ldots \),
\[0 \leq \beta \leq 1\] in Dirichlet’s and Valle-Poussin’s methods.

In the case of Fejer’s method, condition (22) is valid only for \( r = 0, 0 < \beta \leq 1 \). But in the case of Jackson’s and Rogozinski’s method, condition (22) will also be valid if \( r + \beta < 2 \).

**Theorem 5.**

For any square-summable kernel \( k(x, y) \in L^2_p \), and for linear polynomial operator \( U_n(x) \), the following inequality holds:
\[
\zeta_n = \zeta(k; U_n, \phi) = \left\| \frac{\partial}{\partial x} k(x, y) [U_n(\phi(x)) - \phi(y)] \right\|_{L^2_p}
\]  
(23)

\[
\leq \left[ E_{\alpha, \alpha, \alpha}(k) \right]_{L^2_p} \left\| \frac{\partial}{\partial x} k(x, y) \right\|_{L^2_p} + \frac{1}{\pi} \frac{\gamma_n(U_n, \phi)}{\gamma_n(U_n, \phi)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{k(x, y)}{p(x)} \, dx \, dy \leq \left[ E_{\alpha, \alpha, \alpha}(k) \right]_{L^2_p} \left\| \frac{\partial}{\partial x} k(x, y) \right\|_{L^2_p}
\]

for any positive integer \( m \leq n \).

**Proof.** For any function \( \phi(x) \in L^2_p \), with Fourier coefficients \( c_i \) and \( d_i \), and because of \( p(x) \geq 1 \), the following inequality holds:
\[
\left\| c_i \cos(ix) + d_i \sin(ix) \right\|_{L^2_p} \leq \left\| \frac{\partial}{\partial x} \phi(x) \right\|_{L^2_p} \leq \frac{1}{\pi} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x} \phi(x) \right|^2 \, dx \leq \frac{1}{\pi} \int_{\mathbb{R}^d} \left( k(x, y) \right)_{L^2_p} \left( \frac{\partial}{\partial x} \phi(x) \right)_{L^2_p} \, dx \leq \left[ E_{\alpha, \alpha, \alpha}(k) \right]_{L^2_p} \left\| \frac{\partial}{\partial x} k(x, y) \right\|_{L^2_p}
\]

therefore
\[
\left\| c_i \cos(ix) + d_i \sin(ix) \right\|_{L^2_p} \leq \frac{1}{\pi} \int_{\mathbb{R}^d} \left( k(x, y) \right)_{L^2_p} (\phi(x))_{L^2_p} \, dx \leq \left[ E_{\alpha, \alpha, \alpha}(k) \right]_{L^2_p} \left\| \frac{\partial}{\partial x} k(x, y) \right\|_{L^2_p}
\]

Letting
\[
T_{\alpha, \alpha, \alpha}(x, y) = \sum_{i=0}^{m} a_i x_i \cos(y_i) + b_i x_i \sin(y_i),
\]

\[
E_{\alpha, \alpha, \alpha}(k)_{L^2_p} = \left\{ a_i, b_i \right\}_{i=0}^{m} \left\| k(x, y) \right\|_{L^2_p}
\]

and taking into consideration (20) and using Bunyakowskii inequality, we obtain:
\[
\zeta_n = \zeta(k; U_n, \phi) = \left\| \frac{\partial}{\partial x} k(x, y) [U_n(\phi(x)) - \phi(y)] \right\|_{L^2_p}
\]

IV. THE APPROXIMATE SOLUTION

The following theorem shows that for sufficiently good linear methods \( U_n(x) \), the difference between the polynomials \( \varphi_n(x) \) and the original solution \( \varphi(x) \) is sufficiently small.

**Theorem 6.**

If the kernel \( k(x, y) \) of (1) satisfies the assumptions \((A^*)\), all functions appearing in (1) are \( 2\pi \)-periodic in \( x \) and \( y \), and for any linear polynomial operator \( U_n(x) \), if \( |x| R \delta(k; U_n) \approx 1 \) and if equation (1) is replaced by equation (4), the following inequality holds:
\[
\left\| \varphi(x) - \varphi_n(x) \right\|_{L^2_p} \leq (1 + \alpha_n(k)) \left\| \varphi(x) - U_n(\varphi(x)) \right\|_{L^2_p}
\]

in which
\[
\alpha_n(k) = \alpha_n(k) \left[ \delta(k; U_n) \left( \left\| \varphi(x) - U_n(\varphi(x)) \right\|_{L^2_p} \right) \right]
\]

(24)

where \( \delta(k; U_n) \) and \( \zeta(k; U_n, \phi) \) are defined from (8) and (19), respectively, and \( R = \left( 1 + |2| \right) \left\| R(x, y) \right\|_{L^2_p} \), where

\( R(x, y) \) denotes the resolvent of the kernel \( k(x, y) \).

**Proof.** By using equation (1), we represent the solution \( \varphi_n(x) \) of (4) in the form:
\[
\varphi_n(x) = U_n(\varphi(x)) + \lambda U_n \left( \int_{\mathbb{R}^d} k(\cdot, y) \varphi_n(y) \, dy \right) + \int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \, dx \]

and
\[
\lambda U_n \left( \int_{\mathbb{R}^d} k(\cdot, y) \varphi_n(y) \, dy \right) + \int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \, dx = \lambda U_n \left( \int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \right) + \int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \, dx
\]

and
\[
\int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \, dx = \int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \, dx
\]

therefore
\[
\varphi_n(x) = U_n(\varphi(x)) + \lambda U_n \left( \int_{\mathbb{R}^d} k(\cdot, y) \varphi_n(y) \, dy \right) + \int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \, dx
\]

and
\[
\lambda U_n \left( \int_{\mathbb{R}^d} k(\cdot, y) \varphi_n(y) \, dy \right) + \int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \, dx = \lambda U_n \left( \int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \right) + \int_{\mathbb{R}^d} k(\cdot, y) \varphi(y) \, dy \, dx
\]
\[ \|\varphi(x) - \varphi_n(x)\|_{L^2_{\rho(x)}} \leq (1 + a_n(k))(3 + \ln n)E_n^*(\varphi)_{L^2_{\rho(x)}} \]

where \( E_n^*(\varphi)_{L^2_{\rho(x)}} \) is the best approximation to \( \varphi(x) \) by trigonometric polynomial and depends on the smoothness of \( \varphi(x) \) which in turn depends on the smoothness of \( f(x) \) and \( k(x,y) \). As a consequence of (15), (21), (22), (23) and (25), we obtain the following estimate for \( a_n(k) \):

\[ a_n(k) \leq \frac{12\pi(3 + \ln n)\omega_{L^2_{\rho(x)}}(1/n) + E_n^*(\varphi)_{L^2_{\rho(x)}}}{1 - 12\pi(3 + \ln n)\omega_{L^2_{\rho(x)}}(1/n)} \]

\[ \lim a_n(k) = 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad \varphi(x) \in L^2_{\rho(x)}, k(x,y) \in L^2_{\rho(x)}[Q] \]

whenever

\[ \omega_{L^2_{\rho(x)}}(1/n) = o(1/\ln n) \quad \text{or} \quad E_n^*(\varphi)_{L^2_{\rho(x)}} = o(1/\ln n). \]

II) In the case of Vallee-Poisson's method, considering (21), (22) and (24), we obtain:

\[ \|\varphi(x) - \varphi_n(x)\|_{L^2_{\rho(x)}} \leq (1 + a_n(k))(3 + \ln n)E_n^*(\varphi)_{L^2_{\rho(x)}} \]

V. CONCLUSIONS

In conclusion, we note the following statements:

1) It is well-known [5] that one cannot achieve an error less than that corresponding to the best approximation.

The error estimate in (24) with rate of convergence \( a_n(k) \), means that: the rate of convergence of \( \varphi_n(x) \) to \( \varphi(x) \) is comparable with the rate of convergence of the best approximation, which means that the error estimate in (24) is optimal.

2) Applying theorem 6, and also the corresponding results from section 3, we obtain the following results:

I) In the case of the application of Dirichlet's method, from (15) and (24), we have:

\[ \|\varphi(x) - \varphi_n(x)\|_{L^2_{\rho(x)}} \leq (1 + a_n(k))(3 + \ln n)E_n^*(\varphi)_{L^2_{\rho(x)}} \]

where \( E_n^*(\varphi)_{L^2_{\rho(x)}} \) is the best approximation to \( \varphi(x) \) by trigonometric polynomial and depends on the smoothness of \( \varphi(x) \) which in turn depends on the smoothness of \( f(x) \) and \( k(x,y) \). As a consequence of (15), (21), (22), (23) and (25), we obtain the following estimate for \( a_n(k) \):

\[ a_n(k) \leq \frac{12\pi(3 + \ln n)\omega_{L^2_{\rho(x)}}(1/n) + E_n^*(\varphi)_{L^2_{\rho(x)}}}{1 - 12\pi(3 + \ln n)\omega_{L^2_{\rho(x)}}(1/n)} \]

\[ \lim a_n(k) = 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad \varphi(x) \in L^2_{\rho(x)}, k(x,y) \in L^2_{\rho(x)}[Q] \]

\[ \omega_{L^2_{\rho(x)}}(1/n) = o(1/\ln n) \quad \text{or} \quad E_n^*(\varphi)_{L^2_{\rho(x)}} = o(1/\ln n). \]

III) For the methods of Fetté, Jackson and Rogosinski, the quantity \( a_n(k) \) in the relation (24) will not tend to zero for any solution \( \varphi(x) \), but will tend to zero only under the condition that "the solution \( \varphi(x) \), belongs to some subclasses of integrable functions". Restricting ourselves to the holder classes \( W^r H^\beta(L^2_p) \) where \( r > 0 \) is a non-negative integer and \( 0 < \beta \leq 1 \), we obtain the following two cases:

a) In the case of Fetté's method; in order to \( a_n(k) \to 0 \) as \( n \to \infty \) considering (18), (21) and (22), it is sufficient that the following conditions be satisfied:

\[ \varphi(x) \in W^r H^\beta(L^2_p(x)), r > 0, 0 < \beta \leq 1. \]

b) For Rogosinski's and Jackson's methods; in order that \( a_n(k) \to 0 \) as \( n \to \infty \), using (12), (17), (21) and (22), it is then sufficient that the following conditions be satisfied:

\[ \varphi(x) \in W^r H^\beta(L^2_p(x)), r + \beta < 2, 0 < \beta \leq 1. \]

ACKNOWLEDGMENT

We would like to thank and gratitude Prof. Dr. Sayed Shehata El-Ghabaty, Prof. of mathematics, Faculty of science, Benha University, Egypt, and Prof. Dr. Mohamed Irsam Mohamed Hessein, Prof. of Mathematics, Faculty of Engineering (Shoubra), Benha university, Egypt, for their generous help in presenting our work. We are also grateful to anonymous reviewers for helpful comments.
REFERENCES


Dr. Sohier A. Abou Auf, works as a lecturer of Mathematics in Department of Mathematics, Faculty of Science, Benha University, Egypt. The research interests are finding approximate solutions for some integral equations (Fredholm and Volterra).

Mr. Mohamed E. Nasr, works as Assistant lecturer of Mathematics in Department of Mathematics, Faculty of Science, Benha University, Egypt. He has got M.Sc. in approximate solution for some integral equations.