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ScienceDirect
Chaos, Solitons and Fractals 32 (2007) 145–149
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Variational iteration method for one dimensional nonlinear thermoelasticity

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Accepted 10 November 2005

Communicated by Prof. Ji-Huan He

Abstract

This paper applies the variational iteration method to solve the Cauchy problem arising in one dimensional nonlinear thermoelasticity. The advantage of this method is to overcome the difficulty of calculation of Adomian's polynomials in the Adomian's decomposition method. The numerical results of this method are compared with the exact solution of an artificial model to show the efficiency of the method. The approximate solutions show that the variational iteration method is a powerful mathematical tool for solving nonlinear problems.

1. Introduction

In this paper we shall solve a nonlinear system arising in thermoelasticity, the governing equations are ([5,16,18,20], and the references therein):

$$u_{tt} - a(u_x, \theta)u_{xx} + b(u_x, \theta)\theta_x = f(x, t), \quad x \in R^1 \tag{1.1}$$

$$c(u_x, \theta)\theta_t + b(u_x, \theta)u_{xt} - d(\theta)\theta_{xx} = g(x, t), \quad t > 0 \tag{1.2}$$

with initial conditions:

$$u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad \theta(x, 0) = \theta^0(x), \tag{1.3}$$

where $u = u(x, t)$ is the body displacement from equilibrium and $\theta = \theta(x, t)$ is the difference of the body's temperature from a reference $T_0 = 0$, and subscripts denote partial derivatives, a, b, c and d are given smooth functions. The system (1.1)–(1.3) is typically arises in one dimensional nonlinear thermoelasticity, in this case $u(x, t)$ and $\theta(x, t)$ are the displacement and temperature difference, respectively. For more details about the physical meaning of the model see [5,16].

In recent years a great deal of attention has been devoted to study the variational iteration method given by J.H. He for solving a wide range of problems whose mathematical models yield differential equation or system of differential

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0960-0779/$ - see front matter
doi:10.1016/j.chaos.2005.11.028

equations [1–4,6–11,17,19]. For more details about the advantages of the variational iteration method over the Adomian decomposition method see [2].

The paper is organized as follows, in Section 2 we apply the well-known variational iteration method [6–11] for solving the system (1.1) and (1.2). In Section 3 an artificial model is used to show the efficiency of the method and numerical experiments are presented. Finally in Section 4 conclusions are given.

2. The variational iteration method

In this Section, we use the following non-homogeneous, nonlinear system of partial differential equations to illustrate the basic idea of the variational iteration method [6–11].

$$L_1u(x,t) + N_1(u(x,t),\theta(x,t)) = f(x,t), \tag{2.1}$$

$$L_2\theta(x,t) + N_2(u(x,t),\theta(x,t)) = g(x,t), \tag{2.2}$$

where L_1 and L_2 are linear differential operators with respect to time and N_1 and N_2 are nonlinear operators and $f(x,t)$, $g(x,t)$ are given functions. According to the variational iteration method, we can construct a correct functional as follows [6–11]:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\tau)[L_1u_n(x,\tau) + N_1(\tilde{u}_n(x,\tau),\tilde{\theta}_n(x,\tau)) - f(x,\tau)]\,\mathrm{d}\tau, \tag{2.3}$$

$$\theta_{n+1}(x,t) = \theta_n(x,t) + \int_0^t \lambda_2(\tau)[L_2\theta_n(x,\tau) + N_2(\tilde{u}_n(x,\tau),\tilde{\theta}_n(x,\tau)) - g(x,\tau)]\,\mathrm{d}\tau, \tag{2.4}$$

where λ_1 and λ_2 are general Lagrange multipliers, which can be identified optimally via variational theory [12–15]. The second term on the right-hand side in (2.3) and (2.4) are called the correction and the subscript n denotes the nth order approximation. Under a suitable restricted variational assumptions (i.e. $\tilde{u}_n$ and $\tilde{\theta}_n$ are considered as a restricted variation), we can assume that the above correctional functional are stationary (i.e. $\delta u_{n+1} = 0$ and $\delta\theta_{n+1} = 0$), then the Lagrange multipliers can be identified. Now we can start with the given initial approximation and by the above iteration formulas we can obtain the approximate solutions.

3. Numerical example

In this section, we apply the variational iteration method for an artificial model like (1.1) and (1.2) with a, b, c, d, u^0, u^1 and θ^0 defined by

$$a(u_x,\theta) = 2 - u_x\theta, \quad b(u_x,\theta) = 2 + u_x\theta,$$
$$c(u_x,\theta) = 1, \quad d(u_x,\theta) = \theta$$

and $u^0(x) = 1/(1+x^2)$, $u^1(x) = 0$, $\theta^0(x) = 1/(1+x^2)$, and the right-hand side of (1.1) and (1.2) replaced by

$$f(x,t) = \frac{2}{1+x^2} - \frac{2(1+t^2)(3x^2-1)}{(1+x^2)^3}a(w,v) - \frac{2x(1+t)}{(1+x^2)^2}b(w,v),$$
$$g(x,t) = \frac{1}{1+x^2}c(w,v) - \frac{4xt}{(1+x^2)^2}b(w,v) - \frac{2(3x^2-1)(1+t)}{(1+x^2)^3}d(v),$$

where a, b, c and d are defined above and

$$w \equiv w(x,t) = \frac{-2x(1+t^2)}{(1+x^2)^2}, \quad v \equiv v(x,t) = \frac{1+t}{1+x^2},$$

thus the exact solution $u(x,t)$ and $\theta(x,t)$ of the system (1.1) and (1.2) is

$$u(x,t) = \frac{1+t^2}{1+x^2}, \quad \theta(x,t) = \frac{1+t}{1+x^2}.$$

To apply the variational iteration method, we construct the following correction functional:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\tau)\left[u_{n_{tt}} - a\tilde{u}_{n_{xx}} + b\tilde{\theta}_{n_x} - f(x,\tau)\right]\mathrm{d}\tau, \tag{3.1}$$

$$\theta_{n+1}(x,t) = \theta_n(x,t) + \int_0^t \lambda_2(\tau)\left[\theta_{n_t} + \frac{b}{c}\tilde{u}_{n_{xt}} - \frac{d}{c}\tilde{\theta}_{n_{xx}} - \frac{1}{c}g(x,\tau)\right]\mathrm{d}\tau, \tag{3.2}$$

where λ_1 and λ_2 are Lagrange multipliers, $a\tilde{u}_{n_{xx}}, b\tilde{\theta}_{n_x}, \frac{b}{c}\tilde{u}_{n_{xt}}$ and $\frac{d}{c}\tilde{\theta}_{n_{xx}}$ denote the restricted variations (i.e. $\delta a\tilde{u}_{n_{xx}} = \delta b\tilde{\theta}_{n_x} = \delta\frac{b}{c}\tilde{u}_{n_{xt}} = \delta\frac{d}{c}\tilde{\theta}_{n_{xx}} = 0$). Making the above correction functional stationary:

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta\int_0^t \lambda_1(\tau)\left[u_{n_{tt}} - a\tilde{u}_{n_{xx}} + b\tilde{\theta}_{n_x} - f(x,\tau)\right]\mathrm{d}\tau\\ &= \delta u_n(x,t) + (\lambda_1\delta u_{n_t} - \lambda_1'\delta u_n)|_{\tau=t} + \int_0^t\left[\lambda_1''\delta u_n + \lambda_1\left(-a\tilde{u}_{n_{xx}} + b\tilde{\theta}_{n_x} - f(x,\tau)\right)\right]\mathrm{d}\tau = 0,\end{aligned}$$

$$\begin{aligned}\delta\theta_{n+1}(x,t) &= \delta\theta_n(x,t) + \delta\int_0^t \lambda_2(\tau)\left[\theta_{n_t} + \frac{b}{c}\tilde{u}_{n_{xt}} - \frac{d}{c}\tilde{\theta}_{n_{xx}} - \frac{1}{c}g(x,\tau)\right]\mathrm{d}\tau\\ &= \delta\theta_n(x,t) + \lambda_2\delta\theta_n|_{\tau=t} + \int_0^t\left[-\lambda_2'\delta\theta_n + \lambda_2\left(\frac{b}{c}\tilde{u}_{n_{xt}} - \frac{d}{c}\tilde{\theta}_{n_{xx}} - \frac{1}{c}g(x,\tau)\right)\right]\mathrm{d}\tau = 0.\end{aligned}$$

we obtain the following stationary conditions :

$$1 - \lambda_1'(\tau)|_{\tau=t} = 0, \quad \lambda_1''(\tau) = 0, \quad \lambda_1(\tau)|_{\tau=t} = 0, \tag{3.3}$$

$$1 + \lambda_2'(\tau)|_{\tau=t} = 0, \quad \lambda_2'(\tau) = 0. \tag{3.4}$$

The Lagrange multipliers, therefore, can be defined as the following forms:

$$\lambda_1(\tau) = \tau - t, \quad \lambda_2(\tau) = -1. \tag{3.5}$$

Substituting Eq. (3.5) into the correction functional Eqs. (3.1) and (3.2) we obtain the following iteration formulas:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)[u_{n_{tt}} - au_{n_{xx}} + b\theta_{n_x} - f(x,\tau)]\,\mathrm{d}\tau, \tag{3.6}$$

$$\theta_{n+1}(x,t) = \theta_n(x,t) - \int_0^t \left[\theta_{n_t} + \frac{b}{c}u_{n_{xt}} - \frac{d}{c}\theta_{n_{xx}} - \frac{1}{c}g(x,\tau)\right]\mathrm{d}\tau. \tag{3.7}$$

We start with initial approximations $u_0(x,t) = u(x,0)$, and $\theta_0(x,t) = \theta(x,0)$ and by the above formulas, we can obtain $u_1(x,t)$ and $\theta_1(x,t)$ as follows:

$$\begin{aligned}u_0(x,t) &= \frac{1}{1+x^2},\\ u_1(x,t) &= u_0(x,t) + \frac{1}{105(1+x^2)^6}[t^2(105(1+x^2)^5 - 70x^2(-2+x(7+x(-2+6x+4x^3+x^5))))t\\ &\quad - 35(-1+x(-2+x(-2+x(6+4x+8x^3+3x^5))))t^2\\ &\quad + 42x(1+x-3x^2+x^3)t^3 + 14x(1+x-3x^2+x^3)t^4 - 10x(-1+3x^2)t^5)],\\ \theta_0(x,t) &= \frac{1}{1+x^2},\\ \theta_1(x,t) &= \theta_0(x,t) - \left[\frac{-1}{15(1+x^2)^5}(t(30t^3x^2 + 24t^4x^2 + 15(1+x^2)^4\right.\\ &\quad\left. + t^2(10+20x^2-30x^4) - 30t(-1+2x+6x^3+3x^4+6x^5+2x^7)))\right].\end{aligned}$$

Proceeding as the same way, we can obtain $u_2(x,t)$ and $\theta_2(x,t)$, and high order approximations. The numerical results of this example is presented in the following Tables 1–4, we evaluated the numerical results using $n = 2$ terms approximation of the recurrence relations (3.6) and (3.7) at various values of the time t ($t = 0.25$ and $t = 0.5$). Tables 1–4 show the exact solution, the numerical solution and the absolute error. However, many terms can be calculate in order to achieve a high level of accuracy by the variational iteration method.

Table 1
For $t = 0.25$, comparison of the numerical results with exact solution $u(x, t)$ and the absolute error

x	Exact solution	Approximate solution	Absolute error
5	0.0408540	0.0408654	1.14214E−05
6	0.0287105	0.0287162	5.66626E−06
7	0.0212469	0.0212500	3.11183E−06
8	0.0163443	0.0163462	1.84483E−06
9	0.0129562	0.0129573	1.1607E−06
10	0.010519	0.0105198	7.6577E−07
11	0.0087085	0.00870902	5.25173E−07
12	0.00732721	0.00732759	3.71954E−07
13	0.00624973	0.0062500	2.70692E−07
14	0.0053932	0.0053934	2.01627E−07
15	0.00470117	0.00470133	1.53229E−07

Table 2
For $t = 0.25$, comparison of the numerical results with exact solution $\theta(x, t)$ and the absolute error

x	Exact solution	Approximate solution	Absolute error
5	0.0479766	0.0480769	1.013071E−04
6	0.0337349	0.0337838	4.88405E−05
7	0.0249735	0.025000	2.648558E−05
8	0.0192152	0.0192308	1.55626E−05
9	0.0152342	0.0152439	9.72698E−06
10	0.0123699	0.0123762	6.38473E−06
11	0.0102415	0.0102459	4.36097E−06
12	0.00861761	0.00862069	3.07843E−06
13	0.00735071	0.00735294	2.23417E−06
14	0.00634352	0.00634518	1.66026E−06
15	0.00552971	0.00553097	1.25921E−06

Table 3
For $t = 0.5$, Comparison of the numerical results with exact solution $u(x, t)$ and the absolute error

x	Exact solution	Approximate solution	Absolute error
5	0.047890	0.0480769	1.86929E−04
6	0.0336914	0.0337838	9.23846E−05
7	0.0249494	0.025000	5.05943E−05
8	0.0192008	0.0192308	2.9931E−05
9	0.0152251	0.0152439	1.88006E−05
10	0.0123639	0.0123762	1.23875E−05
11	0.0102374	0.0102459	8.48645E−06
12	0.00861468	0.00862069	6.00525E−06
13	0.00734857	0.00735294	4.36714E−06
14	0.00634193	0.00634518	3.25085E−06
15	0.0055285	0.00553097	2.4692E−06

4. Conclusions

In this paper, the variational iteration method [6–11] is applied to solving the Cauchy problem arising in one dimensional nonlinear thermoelasticity with initial conditions, the method needs much less computational work compared with traditional methods. We achieved a very good approximation with the actual solution of the equation by using two terms of the iteration scheme derived above. It is evident that the overall results come very close to the exact solution even using only few terms of the iteration formula. Errors can be made smaller by taking new terms of the iteration formulas. A clear conclusion can be draw from the numerical results that the variational iteration method provides highly accurate numerical solutions without spatial discretizations for nonlinear differential equations. Finally, we point

Table 4
For $t = 0.5$, comparison of the numerical results with exact solution $\theta(x,t)$ and the absolute error

x	Exact solution	Approximate solution	Absolute error
5	0.0568199	0.0576923	8.72439E−04
6	0.0401206	0.0405405	4.19969 E−04
7	0.0297742	0.030000	2.25784 E−04
8	0.0229452	0.0230769	1.31753 E−04
9	0.0182108	0.0182927	8.1886 E−05
10	0.014798	0.0148515	5.35004 E−05
11	0.0122587	0.0122951	3.64006 E−05
12	0.0103192	0.0103448	2.56109 E−05
13	0.00880499	0.00882353	1.85346 E−05
14	0.00760047	0.00761421	1.37398 E−05
15	0.00662677	0.00663717	1.03986 E−05

out that, the approximate solutions $u_i(x,t)$,$\theta_i(x,t)$, $i = 1,2$ are obtained according to the iterative Eqs. (3.6) and (3.7) by using version 5 Mathematica.

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