

Exact solutions of some coupled nonlinear partial differential equations using the homotopy perturbation method

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ABSTRACT

The purpose of this study is to introduce a modification of the homotopy perturbation method using Laplace transform and Padé approximation to obtain closed form solutions of nonlinear coupled systems of partial differential equations. Two test examples are given; the coupled nonlinear system of Burger equations and the coupled nonlinear system in one dimensional thermoelasticity. The results obtained ensure that this modification is capable of solving a large number of nonlinear differential equations that have wide application in physics and engineering.

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1. Introduction

The nonlinear differential equations are generally difficult to solve and their exact solutions are difficult to obtain, therefore, some various approximate methods have recently been developed such as homotopy perturbation method [1–12], and Adomian's decomposition method [13–18] to solve linear and nonlinear differential equations. The homotopy perturbation method has the merits of simplicity and easy execution. Unlike the traditional numerical methods, the HPM does not need, discretization and linearization. Most perturbation methods assume a small parameter exists, but most nonlinear problems have no small parameter at all. Many new methods have been proposed to eliminate the small parameter. Recently, the applications of homotopy theory among scientists appeared, and the homotopy theory becomes a powerful mathematical tool, when it is successfully coupled with perturbation theory. Many authors (see [14], and the references cited therein) are pointed out that the HPM can overcome the difficulties arising in calculation of Adomian's polynomials in Adomian's decomposition method.

In this paper we will present a modification of the homotopy perturbation method, this modification depends on the Padé approximates [19], Laplace transform [20] and Taylor series method [21]. We apply this modification to solve some examples of the coupled nonlinear system of Burger equations and the coupled nonlinear system of 1-D thermo-elasticity. Therefore, this modification of HPM has been widely used in solving nonlinear problems to overcome the shortcoming of other methods. In the next section, we present some basic definitions of the Taylor series method and Padé approximates, which are needed in the next sections.

This paper is arranged in the following manner, in Section 2, we present some basic definitions. In Section 3 we present the standard homotopy perturbation method. In Section 4, we present the modification technique of homotopy perturbation methods. In Section 5, the implementation of this modification on the system of Burger equations is presented. In Section 6, the implementation of this modification on the system of 1-D thermo-elasticity is given, in the end the conclusion is presented in Section 7.

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2. Basic definitions

In this section, we present some basic definitions of the Taylor series method, Padé approximates and the Laplace transform, which needed in the next sections of the paper.

Definition 1. The well-known Taylor series method expands the function $f(x)$ at the point $x = 0$ as:

$$f(x) = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

For example, the Taylor series of the function $f(x) = \sqrt{\frac{1+2x}{1+x}}$ is:

$$f(x) = 1 + \frac{1}{2}x - \frac{5}{8}x^2 + \frac{13}{16}x^3 - \frac{141}{128}x^4 + \dots$$

Remarks. a. To obtain the above series we need computational work in determining the coefficients c_k .
 b. The Padé approximates are a particular type of rational fraction approximation to the value of a function. The idea is to match the Taylor series expansion as far as possible.

For example the PA[1/1] Padé approximant of the function $f(x) = \sqrt{\frac{1+2x}{1+x}}$ is:

$$PA[1/1] = \frac{a + bx}{1 + cx} = \frac{1 + (7/4)x}{1 + (5/4)x}.$$

We now define the Padé approximates [19].

Definition 2. The Padé approximation (PA) of a function $f(x)$ can be written in the form:

$$PA[L/M]_{f(x)}(x) = \frac{\sum_{i=0}^L a_i x^i}{1 + \sum_{i=1}^M b_i x^i}, \tag{1}$$

where L and M are given positive integers. In the above relation there are $L + M + 1$ unknown coefficients. This number suggests that normally the PA[L/M] ought to fit the power series of the function $f(x)$ in the definition (1) through the orders:

$$\sum_{k=0}^{\infty} c_k x^k = PA[L/M]_{f(x)}(x) + o(x^{L+M+1}),$$

returning to the above equation and cross-multiplying, we find that:

$$\begin{aligned} (1 + b_1x + b_2x^2 + \dots + b_Mx^M)(c_0 + c_1x + c_2x^2 + \dots) - (a_0 + a_1x + a_2x^2 + \dots + a_Lx^L) &= o(x^{L+M+1}). \\ a_r = c_r + \sum_{i=1}^r c_{r-i}b_i, \quad r = 0(1)L, \quad (b_i = 0 \text{ if } i > M). \\ c_{L+M-s} + \sum_{i=1}^M c_{L+M-i-s}b_i = 0, \quad s = 0(1)M - 1, \quad (c_i = 0 \text{ if } i < 0) \end{aligned} \tag{2}$$

when this system of $(L + M + 1)$ equations, has a solution, it leads to the desired approximation.

Definition 3. The Laplace transform of a function $g(t)$ is defined to be $\mathcal{L}[g(t)] = \hat{g}(s) = \int_0^{\infty} g(t)e^{-st} dt$.

For example, the Laplace transform of a function $g(t) = e^t \sin(t)$ is $\frac{1}{1+(-1+s)^2}$.

3. Analysis of He's homotopy perturbation method

To illustrate the basic idea of this method [22–24], we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{3}$$

Considering the boundary condition of:

$$B(u, \partial u / \partial n) = 0, \quad r \in \Gamma \tag{4}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω .

The operator A can be divided into two parts of L and N , where L is the linear part, while N is the nonlinear term. Eq. (3) can be rewritten as:

$$L(u) + N(u) - f(r) = 0. \quad (5)$$

Using the homotopy technique, we construct a homotopy as: $v(r, p) : \Omega \times [0, 1] \rightarrow \mathfrak{R}$ which satisfies:

$$H(v, p) = (1 - p)[L(v)] + p[A(v) - f(r)] = 0, \quad r \in \Omega, \quad (6)$$

where $p \in [0, 1]$ is an embedding parameter. Obviously, considering Eq. (6), we will have:

$$H(v, 0) = L(v) = 0 \quad (7)$$

$$H(v, 1) = A(v) - f(r) = 0. \quad (8)$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u(0)$ to $u(r)$. In topology, this is called deformation, and $L(v)$ and $A(v) - f(r)$ are called homotopy. According to HPM, we can first use the embedding parameter p as a “small parameter”, and assume that the solution of Eq. (6) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (9)$$

Setting $p = 1$ results the approximate solution of Eq. (3):

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (10)$$

The combination of the perturbation method and the homotopy method is called HPM, which lacks the limitations of the traditional perturbation methods, although it can have full advantages of the traditional perturbation techniques. The series (10) is convergent for most cases. However, the convergence rate depends on the nonlinear operator, $A(v)$.

The following opinions are suggested by He [5]:

1. The second derivative of $N(v)$ with respect to v must be small because the parameter p may be relatively large, i.e. $p \rightarrow 1$.
2. The norm of $L^{-1}\partial N/\partial v$ must be smaller than one so that the series converges.

4. The modification of homotopy perturbation method

In spite of the advantage of homotopy perturbation method, it has some drawbacks. By using HPM, we get a series, in practice a truncated series solution. The series often coincides with the Taylor expansion of the true solution at point $x = 0$, in the initial value case. Although the series can be rapidly convergent in a very small region, it has very slow convergent rate in the wider region we examine and the truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of the method. All the truncated series solutions have the same problem. Many examples given can be used to support this assertion.

Padé approximant [19] approximates any function by the ratio of two polynomials. The coefficients of the powers occurring in the polynomials are determined by the coefficients in the Taylor expansion. Generally, the Padé approximant can enlarge the convergence domain of the truncated Taylor series, and can improve greatly the convergence rate of the truncated Maclaurin series. So, in this section we will present a modification in the HPM.

The suggested modification of the homotopy perturbation method can be done by using the following algorithm.

Algorithm. Step 1. Solve the differential equation using homotopy perturbation method.

Step 2. Truncated the obtained series solution by using homotopy perturbation method.

Step 3. Take the Laplace transform of the truncated series.

Step 4. Find the Padé approximation of the previous step.

Step 5. Take the inverse Laplace transform.

This modification often gets the closed form of the exact solution of the differential equation.

Now, we will implement this algorithm as an application to illustrate our modification.

5. Implementation of the modification of HPM to a coupled system of Burger equations

Consider the following coupled nonlinear system of Burger equations:

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0, \quad (11)$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0, \quad (12)$$

subject to the following initial conditions

$$u(x, 0) = \sin(x), \quad v(x, 0) = \sin(x). \quad (13)$$

Now, we will solve the system (11)–(13) by using the above algorithm as follows:

Step 1. Solve the system (11)–(13) by using HPM.

According to the HPM, we construct following simple homotopies:

$$u_t + p[-u_{xx} - 2uu_x + (uv)_x] = 0, \tag{14}$$

$$v_t + p[-v_{xx} - 2vv_x + (uv)_x] = 0, \tag{15}$$

where $p \in [0, 1]$ is an embedding parameter, we use it to expand the solutions in the following form:

$$u(x, t) = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \tag{16}$$

$$v(x, t) = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots \tag{17}$$

The approximate solution can be obtained by setting $p = 1$ in Eqs. (16)–(17):

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots \quad \text{and} \quad v(x, t) = v_0 + v_1 + v_2 + v_3 + \dots \tag{18}$$

Now substituting from (16)–(17) into (14)–(15) respectively, and equating the terms with the identical powers of p , we can obtain series of linear equations. These linear equations are easy to solve by using Mathematica software or by setting a computer code to get as many equations as we need in the calculation of the numerical as well as explicit solutions. Here we only write the first few linear equations:

$$p^0 : \dot{u}_0(x, t) = 0, \tag{19a}$$

$$\dot{v}_0(x, t) = 0, \tag{19b}$$

$$p^1 : \dot{u}_1(x, t) = u_{0xx} + 2u_0u_{0x} - (u_0v_{0x} + v_0u_{0x}), \tag{20a}$$

$$\dot{v}_1(x, t) = v_{0xx} + 2v_0v_{0x} - (u_0v_{0x} + v_0u_{0x}), \tag{20b}$$

$$p^2 : \dot{u}_2(x, t) = u_{1xx} + 2(u_0u_{1x} + u_1u_{0x}) - (u_0v_{1x} + u_1v_{0x} + v_0u_{1x} + v_1u_{0x}), \tag{21a}$$

$$\dot{v}_2(x, t) = v_{1xx} + 2(v_0v_{1x} + v_1v_{0x}) - (u_0v_{1x} + u_1v_{0x} + v_0u_{1x} + v_1u_{0x}), \tag{21b}$$

the solutions of Eq. (19) can be calculated by using the following initial conditions:

$$u_0(x, t) = u(x, 0) = \sin(x), \quad v_0(x, t) = v(x, 0) = \sin(x), \tag{22}$$

then, we can derive solutions of (20) in the following form:

$$u_1(x, t) = \int_0^t [u_{0xx} + 2u_0u_{0x} - (u_0v_{0x} + v_0u_{0x})]dt = -t \sin(x), \tag{23}$$

$$v_1(x, t) = \int_0^t [v_{0xx} + 2v_0v_{0x} - (u_0v_{0x} + v_0u_{0x})]dt = -t \sin(x). \tag{24}$$

Also, the solutions of Eq. (22) can be calculated by using the following formula:

$$\begin{aligned} u_2(x, t) &= \int_0^t [u_{1xx} + 2(u_0u_{1x} + u_1u_{0x}) - (u_0v_{1x} + u_1v_{0x} + v_0u_{1x} + v_1u_{0x})]dt \\ &= \frac{t^2}{2} \sin(x), \end{aligned} \tag{25}$$

$$\begin{aligned} v_2(x, t) &= \int_0^t [v_{1xx} + 2(v_0v_{1x} + v_1v_{0x}) - (u_0v_{1x} + u_1v_{0x} + v_0u_{1x} + v_1u_{0x})]dt \\ &= \frac{t^2}{2} \sin(x). \end{aligned} \tag{26}$$

Therefore, the complete approximate solution can be readily obtained by the same iterative process using the Mathematica Package.

Steps 2. Truncate the series solution obtained by HPM

We have applied the method by using the fourth order perturbation only, i.e. the approximate solutions are:

$$\begin{aligned} u(x, t) &\cong U(x, t) = \sum_{i=0}^4 u_i(x, t) = \sin(x) \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} \right), \\ v(x, t) &\cong V(x, t) = \sum_{i=0}^4 v_i(x, t) = \sin(x) \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} \right). \end{aligned} \tag{27}$$

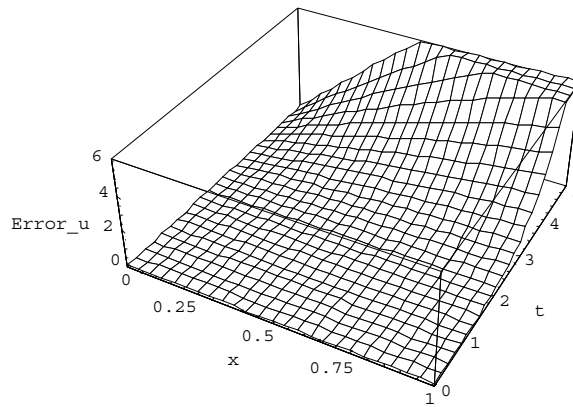


Fig. 1a. The surface error: $error_u = |u(x, t) - U(x, t)|$.

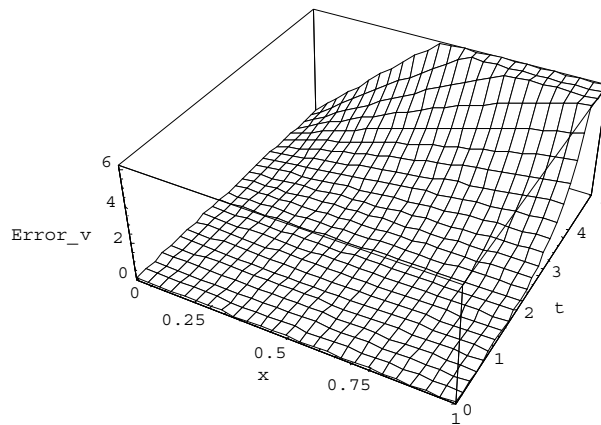


Fig. 1b. The surface error: $error_v = |v(x, t) - V(x, t)|$.

The numerical behavior of the error between the exact solution and the solution obtained by HPM in the regions $0 \leq x \leq 1$ and $0 \leq t \leq 5$ is shown in Fig. 1. The numerical results are obtained by using fourth order perturbation only from the series formulas (18). From this figure, we achieved a very good approximation for the solution of the system at the small values of time t , but at the large values of the time t , the error take large values. It is evident that the overall errors can be made smaller by adding new terms from the iteration formulas.

Step 3. Take the Laplace transform of the equations (27).

$$\begin{aligned} \mathcal{L}[U(x, t)] &= \sin(x) \left[\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} \right], \\ \mathcal{L}[V(x, t)] &= \sin(x) \left[\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} \right]. \end{aligned} \tag{28}$$

For the sake simplicity, let $s = \frac{1}{t}$, then

$$\begin{aligned} \mathcal{L}[U(x, t)] &= \sin(x)[t - t^2 + t^3 - t^4 + t^5], \\ \mathcal{L}[V(x, t)] &= \sin(x)[t - t^2 + t^3 - t^4 + t^5]. \end{aligned} \tag{29}$$

Step 4. Find the Padé approximation of the equations (29).

All of the $\left[\frac{L}{M} \right]$ Padé approximant of the above two equations in (29) with $L > 0, M > 0$ and $L + M < 6$ yields $\left[\frac{L}{M} \right] = \frac{t}{1+t} \sin(x)$.

Replace $t = \frac{1}{s}$, we obtain $\left[\frac{L}{M} \right]$ in terms of s as follows: $\left[\frac{L}{M} \right] = \frac{1}{1+s} \sin(x)$.

Step 5. Take the inverse Laplace transform.

By using the inverse Laplace transform to $[L/M]$, we obtain the true solution;

$$u(x, t) = e^{-t} \sin(x), \quad v(x, t) = e^{-t} \sin(x).$$

The result of the above example show that our method was capable of solving the problems and generates improves homotopy perturbation method in the convergence rate, and that it often close to their exact solutions.

6. Implementation of the modification of HPM to a coupled system of 1-D thermo-elasticity

In this section we will study the numerical and analytical solution of the following nonlinear coupled system in one dimensional thermo-elasticity:

$$u_{tt} - a(u_x, \theta)u_{xx} + b(u_x, \theta)\theta_x = f(x, t), \quad x \in \Omega \tag{30}$$

$$c(u_x, \theta)\theta_t + b(u_x, \theta)u_{tx} - d(\theta)\theta_{xx} = g(x, t), \quad t > 0 \tag{31}$$

with initial conditions:

$$u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad \theta(x, 0) = \theta^0(x). \tag{32}$$

where $u = u(x, t)$ and $\theta = \theta(x, t)$ are the body displacement from equilibrium and the displacement of the body's temperature from a reference $T_0 = 0$, respectively.

Subscripts denote partial derivatives, a, b, c and d are given smooth functions. For more details about the model problem, the regularity of the solution see [25].

Consider the system (30)–(32) with a, b, c, d, u^0, u^1 and θ^0 defined by:

$$a(u_x, \theta) = c(u_x, \theta) = d(\theta) = 1, \quad b(u_x, \theta) = u_x\theta, \\ u^0(x) = \sin(x), \quad u^1(x) = -\sin(x), \quad \theta^0(x) = \cos(x)$$

and the right-hand side:

$$f(x, t) = 2e^{-t} \sin(x) - e^{-3t} \cos^2(x) \sin(x), \\ g(x, t) = -e^{-3t} \cos^3(x).$$

Thus, the exact solution $u(x, t)$ and $\theta(x, t)$ of the system (30)–(32) are:

$$u(x, t) = e^{-t} \sin(x), \quad \theta(x, t) = e^{-t} \cos(x).$$

Therefore, the system take the following form:

$$u_{tt} - u_{xx} + (u_x\theta)\theta_x - f(x, t) = 0, \tag{33}$$

$$\theta_t - \theta_{xx} + (u_x\theta)u_{tx} - g(x, t) = 0, \tag{34}$$

with the following initial conditions:

$$u(x, 0) = \sin(x), \quad u_t(x, 0) = -\sin(x), \quad \theta(x, 0) = \cos(x). \tag{35}$$

Step 1. Solve the system (33)–(35) by using HPM.

According to the homotopy perturbation method, we construct following simple homotopy:

$$u_{tt} + p[-u_{xx} + (u_x\theta)\theta_x - f(x, t)] = 0, \tag{36}$$

$$\theta_t + p[-\theta_{xx} + (u_x\theta)u_{tx} - g(x, t)] = 0, \tag{37}$$

where $p \in [0, 1]$ is an embedding parameter. In case $p = 0$, the Eqs. (36) and (37) becomes linear equations, which are easy to be solved, and when it is one, the Eqs. (36) and (37) turns out to be the original one, Eqs. (33) and (34). In topology, this is called deformation, and $u_{tt}, -u_{xx} + (u_x\theta)\theta_x - f(x, t)$ are called homotopic, also $\theta_t, -\theta_{xx} + (u_x\theta)u_{tx} - g(x, t)$ are called homotopic.

In view of homotopy perturbation method, we use the homotopy parameter p to expand the solutions:

$$u(x, t) = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \tag{38}$$

$$\theta(x, t) = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \dots \tag{39}$$

the approximate solution can be obtained by setting $p = 1$ in Eqs. (38) and (39):

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots \quad \text{and} \quad \theta(x, t) = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \dots \tag{40}$$

Substituting from (38) and (39) into (36) and (37) respectively, and equating the terms with the identical powers of p , we can obtain a series of linear equations. These linear equations are easy to solve by using Mathematica software or by setting a computer code to get as many equations as we need in the calculation of the numerical as well as explicit solutions. Here we only write the first few linear equations:

$$p^0 : \ddot{u}_0(x, t) = 0, \tag{41a}$$

$$\dot{\theta}_0(x, t) = 0, \tag{41b}$$

$$p^1 : \ddot{u}_1(x, t) = f(x, t) + u_{0xx} - (u_{0x}\theta_0)\theta_{0x}, \quad (42a)$$

$$\dot{\theta}_1(x, t) = g(x, t) + \theta_{0xx} - (u_{0x}\theta_0)\dot{u}_{0x}, \quad (42b)$$

$$p^2 : \ddot{u}_2(x, t) = u_{1xx} - (u_{0x}\theta_1 + u_{1x}\theta_0)\theta_{0x} - u_{0x}\theta_0\theta_{1x}, \quad (43a)$$

$$\dot{\theta}_2(x, t) = \theta_{1xx} - (u_{0x}\theta_1 + u_{1x}\theta_0)\dot{u}_{0x} - u_{0x}\theta_0\dot{u}_{1x}, \quad (43b)$$

the solution of Eq. (41) using the initial conditions (35) are:

$$u_0(x, t) = u(x, 0) + u_t(x, 0)t = (1 - t) \sin(x), \quad \theta_0(x, t) = \theta(x, 0) = \cos(x) \quad (44)$$

after substituting from (44) in (42) we can find the solution of (42) in the form:

$$u_1(x, t) = \int_0^t \int_0^t [f(x, t) + u_{0xx} - (u_{0x}\theta_0)\theta_{0x}] dt dt = \left(\frac{1}{2}t^2 + t^3 \left(\frac{\cos(2x)}{6} \right) \right. \\ \left. + t^4 \left(\frac{-5}{48} - \frac{3 \cos(2x)}{16} \right) + t^5 \left(\frac{23}{240} + \frac{9 \cos(2x)}{80} \right) \right) \sin(x) + \dots \quad (45)$$

$$\theta_1(x, t) = \int_0^t [g(x, t) + \theta_{0xx} - (u_{0x}\theta_0)u_{0xt}] dt \\ = -t \cos(x) + \left(t^2 - \frac{3t^3}{2} + \frac{9t^4}{8} + \dots \right) \cos^3(x), \quad (46)$$

also, after substituting by $u_0(x, t)$, $u_1(x, t)$ and $\theta_0(x, t)$, $\theta_1(x, t)$ in (43) we can find the solution of (43),

$$u_2(x, t) = \int_0^t \int_0^t [u_{1xx} - (u_{0x}\theta_1 + u_{1x}\theta_0)\theta_{0x} - u_{0x}\theta_0\theta_{1x}] dt dt \\ = \frac{1}{80640} (t^3(-42(160 + t(-100 + t(120 + t(-89 + 29t)))) \sin(x) + (-6720 + t(9240 \\ + t(-10668 + 83(91 - 31t)t))) \sin(3x) + 3t(560 - 3t(252 + t(-175 + 51t))) \sin(5x)), \quad (47)$$

$$\theta_2(x, t) = \int_0^t [\theta_{1xx} - (u_{0x}\theta_1 + u_{1x}\theta_0)u_{0xt} - u_{0x}\theta_0u_{1xt}] dt \\ = \frac{1}{1920} (t^2 \cos(x)(20(-48 + t(-96 + t(154 + (-96 + t)t))) \cos(2x) \\ + t(2080 + t(-1600 + (978 - 97t)t) + (-160 + 9t(40 + t(-34 + 13t))) \cos(4x))), \quad (48)$$

and so on, ... Therefore, the complete approximate solution can be readily obtained by the same iterative process using the Mathematica Package.

Steps 2. Truncate the series solution obtained by HPM

We have applied the method by using the fourth order perturbation only, i.e. the approximate solutions are:

$$u(x, t) \cong U(x, t) = \sum_{i=0}^4 u_i(x, t) = \sin(x) \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} \right), \quad (49)$$

$$\theta(x, t) \cong \Theta(x, t) = \sum_{i=0}^4 \theta_i(x, t) = \cos(x) \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} \right).$$

Step 3. Take the Laplace transform of the equations (49).

$$\mathcal{L}[U(x, t)] = \sin(x) \left[\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} \right], \quad (50)$$

$$\mathcal{L}[\Theta(x, t)] = \cos(x) \left[\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} \right].$$

For the sake simplicity, let $s = \frac{1}{t}$, then

$$\mathcal{L}[U(x, t)] = \sin(x)[t - t^2 + t^3 - t^4 + t^5], \quad (51)$$

$$\mathcal{L}[\Theta(x, t)] = \cos(x)[t - t^2 + t^3 - t^4 + t^5].$$

Step 4. Find the Padé approximation of the equations (51).

All of the $\left[\frac{L}{M}\right]$ Padé approximant of the above two equations in (51) with $L > 0$, $M > 0$ where $L + M < 6$ yields $\left[\frac{L}{M}\right]_1 = \frac{t}{1+t} \sin(x)$ and $\left[\frac{L}{M}\right]_2 = \frac{t}{1+t} \cos(x)$ respectively.

Replace $t = \frac{1}{s}$, we obtain $\left[\frac{L}{M}\right]$ in terms of s as follows: $\left[\frac{L}{M}\right]_1 = \frac{1}{1+s} \sin(x)$ and $\left[\frac{L}{M}\right]_2 = \frac{1}{1+s} \cos(x)$ respectively.

Step 5. Take the inverse Laplace transform.

By using the inverse Laplace transform to $[L/M]$, we obtain the true solution;

$$u(x, t) = e^{-t} \sin(x), \quad \theta(x, t) = e^{-t} \cos(x). \quad (52)$$

The result of the above example show that our method was capable of solving the problems and generates improves homotopy perturbation method in the convergence rate, and that it often close to their exact solutions.

7. Conclusions

In this paper, the authors presented a modification of the HPM. This modification considerably capable of solving a wide-range class of linear and nonlinear equations; especially the ones of high nonlinearity order in engineering and physics problems. This purpose was satisfied by solving some nonlinear coupled system test examples. The HPM does not need small parameters so that the limitations and non-physical assumptions present in the previous method are eliminated. Therefore, this modification of HPM has widely been used in solving nonlinear problems to overcome the shortcoming of other methods such as Adomian decomposition method.

References

- [1] S. Abbasbandy, Iterated He's homotopy perturbation method for quadratic Riccati differential equation, *Applied Mathematics and Computation* 175 (2006) 581–589.
- [2] D.D. Ganji, Assessment of homotopy–perturbation and perturbation methods in heat radiation equations, *International Communications in Heat and Mass Transfer* 33 (3) (2006) 391–400.
- [3] D.D. Ganji, A. Sadighi, Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (4) (2006) 411–418.
- [4] M. Gorji, D.D. Ganji, S. Soleimani, New application of He's homotopy perturbation method, *International Journal of Nonlinear Sciences and Numerical Simulation* 8 (3) (2007) 319–328.
- [5] J.H. He, A simple perturbation approach to Blasius equation, *Applied Mathematics and Computation* 140 (2–3) (2003) 217–222.
- [6] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, *Chaos, Solitons and Fractals* 26 (2005) 695–700.
- [7] J.H. He, Homotopy perturbation method for solving boundary value problems, *Physics Letters A* 350 (2006) 87–88.
- [8] J.H. He, Homotopy perturbation method for bifurcation of nonlinear problems, *International Journal of Nonlinear Science and Numerical Simulation* 6 (2005) 207–208.
- [9] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, *Applied Mathematics and Computation* 151 (2004) 287–292.
- [10] S. Liao, Comparison between the homotopy analysis method and homotopy perturbation method, *Applied Mathematics and Computation* 169 (2005) 1186–1194.
- [11] M. Rafei, D.D. Ganji, Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (3) (2006) 321–328.
- [12] A. Sadighi, D.D. Ganji, Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation and variational iteration methods, *International Journal of Nonlinear Sciences and Numerical Simulation* 8 (3) (2007) 435–443.
- [13] S. Abbasbandy, M.T. Darvishi, A numerical solution of Burger's equation by modified Adomian method, *Applied Mathematics and Computation* 163 (2005) 1265–1272.
- [14] S. Abbasbandy, A numerical solution of Blasius equation by Adomian's decomposition method and comparison with homotopy perturbation method, *Chaos, Solitons and Fractals* 31 (2007) 257–260.
- [15] G. Adomian, A review of the decomposition method in applied mathematics, *Mathematical Analysis and Applications* 135 (1988) 501–544.
- [16] G. Adomian, *Nonlinear Stochastic Systems and Applications to Physics*, Kluwer Academic Publishers, Dordrecht, 1989.
- [17] E. Babolian, J. Biazar, On the order of convergence of Adomian method, *Applied Mathematics and Computation* 130 (2002) 383–387.
- [18] S.M. El-Sayed, D. Kaya, On the numerical solution of the system of two dimensional Burger's equations by the decomposition method, *Applied Mathematics and Computation* 158 (2004) 101–109.
- [19] G.A. Barker Jr., *Essentials of Padé Approximants*, Academic Press, 1975.
- [20] R.S. Murray, Laplace Transform, in: *Schaum's Outline Series*, New York, 1981.
- [21] A.M. Wazwaz, A comparison between Adomian decomposition method and Taylor series method in the series solution, *Applied Mathematics and Computation* 97 (1998) 37–44.
- [22] N.H. Sweilam, M.M. Khader, R.F. Al-Bar, Numerical studies for a multi-order fractional differential equation, *Physics Letters A* 371 (2007) 26–33.
- [23] H. Tari, D.D. Ganji, M. Rostamian, Approximate solutions of K (2, 2), KdV and modified KdV equations by variational iteration method, homotopy perturbation method and homotopy analysis method, *International Journal of Nonlinear Sciences and Numerical Simulation* 8 (2) (2007) 203–210.
- [24] E. Yusufoglu, Homotopy perturbation method for solving a nonlinear system of second order boundary value problems, *International Journal of Nonlinear Sciences and Numerical Simulation* 8 (3) (2007) 353–358.
- [25] N.H. Sweilam, M.M. Khader, Variational iteration method for one dimensional nonlinear thermo-elasticity, *Chaos, Solitons and Fractals* 32 (2007) 145–149.