

HOMOTOPY PERTURBATION METHOD FOR MULTI-DIMENSIONAL NONLINEAR COUPLED SYSTEM OF PARABOLIC AND HYPERBOLIC EQUATIONS

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ABSTRACT. In this paper, the homotopy perturbation method (HPM) proposed by J. H. He is adopted for solving multi-dimensional nonlinear coupled system of parabolic and hyperbolic equations. The numerical results of the present method are compared with the exact solution of an artificial multi-dimensional nonlinear coupled system of parabolic and hyperbolic model to show the efficiency of the method. Moreover, comparison are made between the results obtained by the present method and that obtained by Adomian decomposition method (ADM). It is found that the present method works extremely well, very efficient, simple and convenient.

1. Introduction

In the past few years, a growing interest towards the applications of the homotopy technique in nonlinear problems which can be described by weakly (or strongly) nonlinear partial differential equations. The HPM (see [1], [4], [6]–[9], [12], [16]) is one of the methods which has received much concern, it has the merits of simplicity and easy execution. Unlike the traditional numerical methods [10], the HPM does not need, discretization, linearization. Many authors (see [2], [3], [5], [15], [17], [18] and the references cited therein) are pointed out

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that the HPM can overcome the difficulties arising in calculation of Adomian's polynomials in ADM.

The multi-dimensional coupled systems of parabolic and hyperbolic equations often appear in the study of circled fuel reactor, high-temperature hydrodynamics and thermo-elasticity problems, see [14] and the references cited therein. For some kinds of nonlinear thermo-elasticity coupled systems, there are several publications (see [10], [11], [19] and the references cited therein) studying the numerical computations, existence and smooth properties of their solutions. In this paper we will use the merits of simplicity of HPM to solve the following multi-dimensional coupled system of non-linear partial differential equations (see [14]).

$$(1.1) \quad u_t - \nabla \cdot (\alpha(X, t, u, v) \nabla u) = f(X, t, u, v, u_x, v_x, u_y, v_y),$$

$$(1.2) \quad v_{tt} - \nabla \cdot (\beta(X, t, u, v) \nabla v) = g(X, t, u, v, u_x, v_x, u_y, v_y, u_t, v_t).$$

Here $X = (x, y)$, $X \in \Omega = [0, d_1] \times [0, d_2]$, $t \in [0, T]$ with the following boundary and initial conditions:

$$(1.3) \quad \begin{aligned} u(X, t) = v(X, t) = 0, \quad X \in \partial\Omega, \quad t \in [0, T], \\ u(X, 0) = u_0(X), \quad v(X, 0) = v_0(X), \quad v_t(X, 0) = v_1(X), \quad X \in \Omega, \end{aligned}$$

where $\alpha, \beta, f, g, u_0, v_0, v_1$ are known functions. For more details on such model see [14] and the references cited therein.

2. Analysis and Implementation of the HPM

To illustrate the basic idea of HPM, let us consider the following system of partial differential equations:

$$(2.1) \quad L_1 u(X, t) + N_1(u(X, t), v(X, t)) - f(X, t) = 0,$$

$$(2.2) \quad L_2 v(X, t) + N_2(u(X, t), v(X, t)) - g(X, t) = 0,$$

where L_1, L_2 are linear operators and N_1, N_2 are nonlinear operators. With a suitable initial and boundary conditions, f and g are known analytic functions. By the homotopy technique, we can construct homotopies $\tilde{u}(X, p): \Omega \times [0, 1] \rightarrow R$ and $\tilde{v}(X, p): \Omega \times [0, 1] \rightarrow R$ such that:

$$(2.3) \quad H_1(\tilde{u}, p) = L_1(\tilde{u}) - L_1(u_0) + p L_1(u_0) + p [N_1(\tilde{u}, \tilde{v}) - f(X, t)] = 0,$$

$$(2.4) \quad H_2(\tilde{v}, p) = L_2(\tilde{v}) - L_2(v_0) + p L_2(v_0) + p [N_2(\tilde{u}, \tilde{v}) - g(X, t)] = 0,$$

where p is an embedding parameter, u_0 and v_0 are initial approximation of the solutions. It is obvious that when $p = 0$, equations (2.3) and (2.4) are linear equations and when $p = 1$, it becomes the original non-linear equations. The embedding parameter monotonically increases from zero to unit as the trivial problems. $L_1(\tilde{u}) - L_1(u_0) = 0$, $L_2(\tilde{v}) - L_2(v_0) = 0$, are continuously deform

to the problem (2.3) and (2.4), respectively. This is a basic idea of homotopy method which is continuously deform a simple problem easy to solve into the difficult problem under study. In view of HPM, we use the homotopy parameter to expand the solutions

$$(2.5) \quad \tilde{u}(X, t) = \tilde{u}_0 + p\tilde{u}_1 + p^2\tilde{u}_2 + p^3\tilde{u}_3 + \dots,$$

$$(2.6) \quad \tilde{v}(X, t) = \tilde{v}_0 + p\tilde{v}_1 + p^2\tilde{v}_2 + p^3\tilde{v}_3 + \dots$$

The approximate solutions can be obtained by setting $p = 1$ in (2.5) and (2.6):

$$u(X, t) = \lim_{p \rightarrow 1} \tilde{u} = \tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + \dots,$$

$$v(X, t) = \lim_{p \rightarrow 1} \tilde{v} = \tilde{v}_0 + \tilde{v}_1 + \tilde{v}_2 + \dots$$

Now, in this section, we apply the HPM to an artificial model like in (1.1)–(1.3) in order to demonstrate the high order accuracy and to compare the HPM solution with the exact solution. Let us consider the two dimensional nonlinear coupled system (1.1)–(1.3) with the following coefficients and functions:

$$\begin{aligned} \alpha(x, y, t, u, v) &= u - 2v, & \beta(x, y, t, u, v) &= v - 2u, \\ f(x, y, t) &= 2t - 12x^2 - 12y^2 + 4(t^2 - x^2 - y^2 - 2(t^2 + x^2 + y^2)), \\ g(x, y, t) &= 2 - 12x^2 - 12y^2 - 4(t^2 + x^2 + y^2 - 2(t^2 - x^2 - y^2)), \end{aligned}$$

and the following initial conditions:

$$u(x, y, 0) = -(x^2 + y^2), \quad v(x, y, 0) = x^2 + y^2, \quad v_t(x, y, 0) = 0, \quad \Omega = [0, 1] \times [0, 1].$$

In this case, the coupled system of parabolic and hyperbolic equations has the following exact solution:

$$u(x, y, t) = t^2 - (x^2 + y^2), \quad v(x, y, t) = t^2 + (x^2 + y^2).$$

Now, the model problem (1.1)–(1.2) can be written in the following operator form:

$$(2.7) \quad L_t u - N_1(u, v) - f(X, t) = 0,$$

$$(2.8) \quad L_{tt} v - N_2(u, v) - g(X, t) = 0,$$

where the notations $L_t = \partial/\partial t$ and $L_{tt} = \partial^2/\partial t^2$ symbolize the linear differential operators and the nonlinear operators $N_1(u, v)$ and $N_2(u, v)$ are defined by:

$$(2.9) \quad N_1(u, v) = (u_x - 2v_x)u_x + (u_y - 2v_y)u_y + (u - 2v)(u_{xx} + u_{yy}),$$

$$(2.10) \quad N_2(u, v) = (v_x - 2u_x)v_x + (v_y - 2u_y)v_y + (v - 2u)(v_{xx} + v_{yy}).$$

According to the HPM, we construct the following simple homotopy

$$(2.11) \quad H_1(\tilde{u}, p) = L_t(\tilde{u}) - L_t(\tilde{u}_0) + p L_t(\tilde{u}_0) - p [N_1(\tilde{u}, \tilde{v}) + f(X, t)] = 0,$$

$$(2.12) \quad H_2(\tilde{v}, p) = L_{tt}(\tilde{v}) - L_{tt}(\tilde{v}_0) + p L_{tt}(\tilde{v}_0) - p [N_2(\tilde{u}, \tilde{v}) + g(X, t)] = 0,$$

where $p \in [0, 1]$ is an embedding parameter. It is obvious that when $p = 0$, the above equations become linear equations of the form $L_t(\tilde{u}) = L_t(\tilde{u}_0)$, $L_{tt}(\tilde{v}) = L_{tt}(\tilde{v}_0)$, and it turns to the original equations when $p = 1$. The HPM uses the homotopy parameter as expanding parameter to obtain

$$\begin{aligned} u(X, t) &= \lim_{p \rightarrow 1} \tilde{u} = \tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + \dots, \\ v(X, t) &= \lim_{p \rightarrow 1} \tilde{v} = \tilde{v}_0 + \tilde{v}_1 + \tilde{v}_2 + \dots \end{aligned}$$

Substituting from (2.5) and (2.6) into (2.11) and (2.12), respectively and using equations (2.9) and (2.10), and equating the terms with the identical powers of p , we can obtain a series of linear equations. These linear equations are easy to solve by using Mathematica software. Here we only write the first few linear equations and for simplicity, we will use the notion u_i , v_i for the approximate solution instead of \tilde{u}_i , \tilde{v}_i :

$$(2.13) \quad \dot{u}_0(X, t) = 0, \quad \ddot{v}_0(X, t) = 0,$$

$$(2.14) \quad \begin{aligned} \dot{u}_1(X, t) &= f(X, t) + (u_{0x} - 2v_{0x})u_{0x} \\ &\quad + (u_{0y} - 2v_{0y})u_{0y} + (u_0 - 2v_0)(u_{0xx} + u_{0yy}), \end{aligned}$$

$$(2.15) \quad \begin{aligned} \ddot{v}_1(X, t) &= g(X, t) + (v_{0x} - 2u_{0x})v_{0x} \\ &\quad + (v_{0y} - 2u_{0y})v_{0y} + (v_0 - 2u_0)(v_{0xx} + v_{0yy}), \end{aligned}$$

$$(2.16) \quad \begin{aligned} \dot{u}_2(X, t) &= (u_{1x} - 2v_{1x})u_{0x} + (u_{0x} - 2v_{0x})u_{1x} \\ &\quad + (u_{1y} - 2v_{1y})u_{0y} + (u_{0y} - 2v_{0y})u_{1y} \\ &\quad + (u_0 - 2v_0)(u_{1xx} + u_{1yy}) + (u_1 - 2v_1)(u_{0xx} + u_{0yy}), \end{aligned}$$

$$(2.17) \quad \begin{aligned} \ddot{v}_2(X, t) &= (v_{1x} - 2u_{1x})v_{0x} + (v_{0x} - 2u_{0x})v_{1x} \\ &\quad + (v_{1y} - 2u_{1y})v_{0y} + (v_{0y} - 2u_{0y})v_{1y} \\ &\quad + (v_0 - 2u_0)(v_{1xx} + v_{1yy}) + (v_1 - 2u_1)(v_{0xx} + v_{0yy}), \end{aligned}$$

the solution of equation (2.13) using the initial conditions are:

$$\begin{aligned} u_0(x, y, t) &= u(x, y, 0) = -(x^2 + y^2), \\ v_0(x, y, t) &= v(x, y, 0) + v_t(x, y, 0)t = x^2 + y^2, \end{aligned}$$

after substituting by $u_0(x, y, t)$ and $v_0(x, y, t)$ in (2.14)–(2.15), we can find the solution of (2.14)–(2.15) in the form:

$$\begin{aligned} u_1(x, y, t) &= \int_0^t [(u_{0x} - 2v_{0x})u_{0x} + (u_{0y} - 2v_{0y})u_{0y} \\ &\quad + (u_0 - 2v_0)(u_{0xx} + u_{0yy}) + f(X, \tau)] d\tau = t^2 - \frac{4}{3}t^3, \\ v_1(x, y, t) &= \int_0^t \int_0^t [(v_{0x} - 2u_{0x})v_{0x} + (v_{0y} - 2u_{0y})v_{0y} \\ &\quad + (v_0 - 2u_0)(v_{0xx} + v_{0yy}) + g(X, \tau)] d\tau d\tau = t^2 + \frac{1}{3}t^4, \end{aligned}$$

after substituting by u_0, u_1 and v_0, v_1 in (2.16)–(2.17), we can find the solution of (2.16)–(2.17) in the form:

$$\begin{aligned} u_2 &= \int_0^t [(u_{1x} - 2v_{1x})u_{0x} + (u_{0x} - 2v_{0x})u_{1x} + (u_{1y} - 2v_{1y})u_{0y} \\ &\quad + (u_{0y} - 2v_{0y})u_{1y} + (u_0 - 2v_0)(u_{1xx} + u_{1yy}) \\ &\quad + (u_1 - 2v_1)(u_{0xx} + u_{0yy})] d\tau = \frac{4}{15}t^3(5 + 5t + 2t^2), \\ v_2 &= \int_0^t \int_0^t [(v_{1x} - 2u_{1x})v_{0x} + (v_{0x} - 2u_{0x})v_{1x} + (v_{1y} - 2u_{1y})v_{0y} \\ &\quad + (v_{0y} - 2u_{0y})v_{1y} + (v_0 - 2u_0)(v_{1xx} + v_{1yy}) \\ &\quad + (v_1 - 2u_1)(v_{0xx} + v_{0yy})] d\tau d\tau = \frac{t^4}{45}(-15 + 24t + 2t^2). \end{aligned}$$

Also, we can find the solutions $u_3(x, y, t)$, $v_3(x, y, t)$ in the form:

$$\begin{aligned} u_3(x, y, t) &= \frac{-4t^4}{315}(105 + 126t - 28t^2 - 4t^3), \\ v_3(x, y, t) &= \frac{-t^5}{315}(168 + 126t + 16t^2 - t^3). \end{aligned}$$

Proceeding in the same way, we can obtain high order approximations. In order to illustrate the advantages and the accuracy of the HPM for solving the present problem, we calculate the fifteen order perturbation, i.e. the approximate solutions are:

$$(2.18) \quad u(x, y, t) = u_0 + \dots + u_{15} \quad \text{and} \quad v(x, y, t) = v_0 + \dots + v_{15},$$

and compare it with the exact solution, where $d_1 = d_2 = 1$, $T = 2$. The numerical results are shown in Table 1. We achieved a very good approximation with the actual solution of the equations. It is evident that even using few terms of the series, the overall results getting very close to exact solution, errors can be made smaller by adding new terms of the expand series. From Table 1, we

t	$\ u - u_{15}\ _2$	$\ v - v_{15}\ _2$
0.25	2.77334E-32	1.92593E-34
0.50	6.87141E-23	3.30001E-25
0.75	4.28078E-18	7.90194E-20
1.00	1.78134E-16	3.51264E-17
1.25	4.85668E-12	1.18268E-13
1.50	4.30891E-12	3.11003E-11
1.75	2.64388E-07	2.70213E-09
2.00	2.13518E-05	2.19768E-06

TABLE 1. The L^2 -norm error of u, v at different times

can conclude that the HPM scheme has a very high accuracy comparing with the exact solution even for long time period.

3. Analysis and Implementation of the ADM

To explain and implement the ADM to the same model, we will consider the system (2.7)–(2.8) with respect to (2.9) and (2.10). By using the inverse operators, we can write the system (2.7)–(2.8) in the following form:

$$(3.1) \quad u(x, y, t) = u(x, y, 0) + L_t^{-1} f(x, y, t) + L_t^{-1} N_1(u, v),$$

$$(3.2) \quad v(x, y, t) = v(x, y, 0) + v_t(x, y, 0)t + L_{tt}^{-1} g(x, y, t) + L_{tt}^{-1} N_2(u, v),$$

where the inverse operators are defined by

$$L_t^{-1} = \int_0^t (\cdot) dt, \quad L_{tt}^{-1} = \int_0^t \int_0^t (\cdot) dt dt.$$

The ADM, suggests that the solution $u(x, y, t)$ and $v(x, y, t)$ can be decomposed by an infinite series of components:

$$u(x, y, t) = \sum_{i=0}^{\infty} U_i(x, y, t), \quad v(x, y, t) = \sum_{i=0}^{\infty} V_i(x, y, t),$$

and the nonlinear terms defined in (2.9) and (2.10) decomposed by the infinite series:

$$N_k(u, v) = \sum_{i=0}^{\infty} A_{ki}, \quad k = 1, 2,$$

where $U_i(x, y, t)$ and $V_i(x, y, t)$, $i \geq 0$, are the components of $u(x, y, t)$, $v(x, y, t)$ that will be elegantly determined and are called Adomian's polynomials and definite by

$$(3.3) \quad A_{kn} = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N_k \left(\sum_{j=0}^n \lambda^j u_j, \sum_{j=0}^n \lambda^j v_j \right) \right]_{\lambda=0}, \quad n \geq 0,$$

from the above considerations, the decomposition method defines the components U_i and V_i for, $i \geq 0$ by the following recursive relationships

$$(3.4) \quad \begin{aligned} U_0(X, t) &= u(X, 0), \\ U_1(X, t) &= L_t^{-1}[f(X, t) + A_{10}], \\ U_{n+1}(X, t) &= L_t^{-1}[A_{1n}], \quad n \geq 1, \end{aligned}$$

$$(3.5) \quad \begin{aligned} V_0(X, t) &= v(X, 0) + v_t(X, 0)t, \\ V_1(X, t) &= L_{tt}^{-1}[g(X, t) + A_{20}], \\ V_{n+1}(X, t) &= L_{tt}^{-1}[A_{2n}], \quad n \geq 1. \end{aligned}$$

This will enable us to determine the components U_n and V_n recurrently. However, in many cases the exact solution in a closed form may be obtained. For numerical comparisons purpose, we construct the solutions $u(x, y, t)$ and $v(x, y, t)$ such that:

$$\lim_{n \rightarrow \infty} \Psi_n = u(x, y, t), \quad \lim_{n \rightarrow \infty} \Theta_n = v(x, y, t),$$

where

$$\Psi_n(x, y, t) = \sum_{i=0}^{n-1} U_i(x, y, t), \quad \Theta_n(x, y, t) = \sum_{i=0}^{n-1} V_i(x, y, t), \quad n \geq 0.$$

In an algorithmic form, the ADM can be implemented the coupled solutions as follows:

ALGORITHM. Let n be the iteration index, set a suitable value for the tolerance (Tol.)

Step 1. Compute the initial approximations

$$\begin{aligned} U_0(X, t) &= u(X, 0), & U_1(X, t) &= L_t^{-1}(f(X, t) + A_{10}), \\ V_0(X, t) &= v(X, 0) + v_t(X, 0)t, & V_1(X, t) &= L_{tt}^{-1}(g(X, t) + A_{20}) \end{aligned}$$

with respect to (1.3), set $n = 1$.

Step 2. Compute the Adomian's polynomials A_{1n} and A_{2n} from (3.3).

Step 3. Use the calculated values of U_n and V_n to compute U_{n+1} from (3.4).

Step 4. Define $U_n := U_{n+1}$.

Step 5. Use the calculated values of U_n and V_n to compute V_{n+1} from (3.5).

Step 6. If $\max_{X \in \Omega} |U_{n+1} - U_n| < \text{Tol.}$ and $\max_{X \in \Omega} |V_{n+1} - V_n| < \text{Tol.}$ stop otherwise continue.

Step 7. Set $U_{n+1} := U_n$.

Step 8. Set $n = n + 1$ and return to Step 2.

To find the solution of the system (1.1)–(1.3) using ADM, we can give the first Adomian's polynomials of the A_{ki} using equations (24) as follows:

$$\begin{aligned}
A_{10} &= (u_{0x} - 2v_{0x})u_{0x} + (u_{0y} - 2v_{0y})u_{0y} + (u_0 - 2v_0)(u_{0xx} + u_{0yy}), \\
A_{20} &= (v_{0x} - 2u_{0x})v_{0x} + (v_{0y} - 2u_{0y})v_{0y} + (v_0 - 2u_0)(v_{0xx} + v_{0yy}), \\
A_{11} &= (u_{1x} - 2v_{1x})u_{0x} + (u_{0x} - 2v_{0x})u_{1x} + (u_{1y} - 2v_{1y})u_{0y} \\
&\quad + (u_{0y} - 2v_{0y})u_{1y} + (u_0 - 2v_0)(u_{1xx} + u_{1yy}) + (u_1 - 2v_1)(u_{0xx} + u_{0yy}), \\
A_{21} &= (v_{1x} - 2u_{1x})v_{0x} + (v_{0x} - 2u_{0x})v_{1x} + (v_{1y} - 2u_{1y})v_{0y} \\
&\quad + (v_{0y} - 2u_{0y})v_{1y} + (v_0 - 2u_0)(v_{1xx} + v_{1yy}) + (v_1 - 2u_1)(v_{0xx} + v_{0yy}).
\end{aligned}$$

Proceeding as the same way, the above Adomian's polynomials of the A_{1k} and A_{2k} can be evaluated. Now, by using the given initial conditions and the above choice of the coefficient and functions we can derive:

$$\begin{aligned}
U_1(x, y, t) &= t^2 - \frac{4}{3}t^3, \\
U_2(x, y, t) &= \frac{4}{15}t^3(5 + 5t + 2t^2), \\
U_3(x, y, t) &= \frac{-4t^4}{315}(105 + 126t - 28t^2 - 4t^3), \\
U_4 &= \frac{8t^5}{315}(378 + 126t - 261t^2 + t^3), \\
V_1(x, y, t) &= t^2 + \frac{1}{3}t^4, \\
V_2(x, y, t) &= \frac{t^4}{45}(-15 + 24t + 2t^2), \\
V_3(x, y, t) &= \frac{-t^5}{315}(168 + 126t + 16t^2 - t^3), \\
V_4 &= \frac{t^6}{14175}(5040 + 3600t - 1245t^2 + 2t^3),
\end{aligned}$$

Proceeding as the same way, we can obtain high order approximation. In order to verify numerically whether the proposed methodology lead to higher accuracy, we evaluate the numerical solutions using the n -term approximation. It is to be noted that Ψ_n and Θ_n show clearly the convergence to the correct limit are presented. Although, we have difficulties to calculate the Adomian's polynomials, but we can arrive to the same order of accuracy of the solutions using $n = 15$ terms of the decomposition series derived above (3.4)–(3.5) where $d_1 = d_2 = 1$, $T = 2$.

4. Conclusions

In this paper, HPM is used to solve numerically, the multi-dimensional non-linear coupled system of parabolic and hyperbolic equations and compared with

ADM. The present method has some obvious merits: (1) the mathematical calculations of the approximate solution is simpler than the other methods; (2) the solution obtained by the present method has a very high accuracy comparing with the exact solution even for long time period; (3) the method needs not to calculate Adomian's polynomials. (4) the HPM is highly accurate numerical solutions without spatial discretizations or linearization for nonlinear partial differential equations. Finally, we point out that the corresponding analytical and numerical solutions are obtained according to the iteration equations using Mathematica 5.

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