On a nonlinear delay population model

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Abstract

The nonlinear delay differential equation $\dot{x}(t) = r(t)[g(t, x_t) - h(x(t))],\ t \geq 0$ is considered. Sufficient conditions are established for the uniform permanence of the positive solutions of the equation. In several particular cases, explicit formulas are given for the upper and lower limit of the solutions. In some special cases, we give conditions which imply that all solutions have the same asymptotic behavior, in particular, when they converge to a periodic or constant steady-state.

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1. Introduction

The scalar nonautonomous differential equation

$$\dot{N}(t) = a(t)N(t) - r(t)N^2(t), \qquad t \geq 0 \tag{1.1}$$

is known as the logistic equation in mathematical ecology. Eq. (1.1) is a prototype in modeling the dynamics of single species population systems whose biomass or density is denoted by a function N of the time variable. The functions $a(t)$ and $r(t)$ are time dependent net birth and self-inhibition rate functions, respectively. The carrying capatICY of the habitat is the time dependent function

$$K(t) = \frac{a(t)}{r(t)}, \qquad t \geq 0. \tag{1.2}$$

By using this notation, Eq. (1.1) can be written as

$$\dot{N}(t) = r(t)\Big(K(t)N(t) - N^2(t)\Big), \qquad t \geq 0, \tag{1.3}$$

or

$$\dot{N}(t) = r(t)\Big(K_0 N(t) - N^2(t)\Big), \qquad t \geq 0 \tag{1.4}$$

whenever the carrying capacity is constant, i.e., $K(t) = K_0$, $t \geq 0$ with a $K_0 > 0$.

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It follows by elementary techniques that the above equations with the initial condition

$$N(0) = N_0 > 0 \tag{1.5}$$

has a unique solution $N(N_0)(t)$ of the initial value problem (IVP) (1.4) and (1.5) given by the explicit formula

$$N(N_0)(t) = \frac{N_0 K_0 e^{K_0 \int_0^t r(s)\, ds}}{K_0 + N_0(e^{K_0 \int_0^t r(s)\, ds} - 1)}, \qquad t \geq 0. \tag{1.6}$$

From the above formula, we get that either

$$\int_0^\infty r(s)\, ds = \infty \tag{1.7}$$

and

$$N(N_0)(\infty) := \lim_{t\to\infty} N(t) = K_0 \qquad \text{for any } N_0 > 0,$$

or

$$\int_0^\infty r(s)\, ds < \infty \tag{1.8}$$

and

$$N(N_0)(\infty) = \frac{N_0 K_0 e^{K_0 \int_0^\infty r(s)\, ds}}{K_0 + N_0(e^{K_0 \int_0^\infty r(s)\, ds} - 1)} \neq K_0 \qquad \text{for any } N_0 \neq K_0.$$

Thus K_0 is a global attractor of (1.4) with respect to the positive solutions if and only (1.7) holds.

It follows by some elementary technique that for any $N_0 > 0$ the solution $N(N_0)(t)$ of the IVP (1.3) and (1.5) obeys

$$\underline{K}(\infty) \leq \liminf_{t\to\infty} N(N_0)(t) \leq \limsup_{t\to\infty} N(N_0)(t) \leq \overline{K}(\infty) \tag{1.9}$$

for any $N_0 > 0$, if

$$0 < \underline{K}(\infty) := \liminf_{t\to\infty} K(t) \leq \limsup_{t\to\infty} K(t) =: \overline{K}(\infty) < \infty \tag{1.10}$$

and (1.7) holds. Motivated by the above simple results, in this paper we give lower and upper estimations for the positive solutions of the nonlinear delay differential equation

$$\dot{x}(t) = r(t)\Big(g(t, x_t) - h(x(t))\Big), \qquad t \geq 0, \tag{1.11}$$

where $x_t(\theta) = x(t+\theta)$, $-\tau \leq \theta \leq 0$, $r, h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $g \in C(\mathbb{R}_+ \times C, \mathbb{R}_+)$. Here $\tau > 0$ is fixed, $\mathbb{R}_+ := [0, \infty)$ and $C := C([-\tau, 0], \mathbb{R})$. Eq. (1.11) can be considered as a population model equation with delay in the birth term $r(t)g(t, x_t)$, and no delay in the self-inhibition term $r(t)h(x(t))$. The form of the delay is based on the works of the authors [3], [5], [8], [10], [11], [12], [14], [15], [16], [17], who have argued that the delay should enter in the birth term rather than in death of inhibition term. Eq. (1.11) includes, e.g., the next equations

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t) x(t - \tau_k(t)) - \beta(t) x^2(t), \qquad t \geq 0, \tag{1.12}$$

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t) x^p(t - \tau_k(t)) - \beta(t) x^q(t), \qquad t \ge 0, \quad 0 < p < q, \; q \ge 1, \tag{1.13}$$

$$\dot{x}(t) = \alpha(t) f(x(t-\tau)) - \beta(t) g(x(t)), \qquad t \ge 0, \tag{1.14}$$

and

$$\dot{x}(t) = \frac{\alpha(t)x(t-\tau)}{1+\gamma(t)x(t-\tau)} - \beta(t)x^2(t), \qquad t \ge 0 \tag{1.15}$$

with discrete delays, or

$$\dot{x}(t) = \alpha(t) \int_{-\tau}^{0} f(s, x(t+s))\, ds - \beta(t) g(x(t)), \qquad t \ge 0 \tag{1.16}$$

with distributed delay.

Recently, lower and upper estimations of the positive solutions of Eq. (1.12) were proved in [2] and [6] under the assumptions that the coefficients α_k and β satisfy

$$\alpha_0 \le \alpha_k(t) \le A_0, \qquad \beta_0 \le \beta(t) \le B_0, \qquad t \ge 0, \; k = 1, \ldots, n \tag{1.17}$$

with some positive constants α_0, A_0, β_0 and B_0. The following theorem, which is a consequence of our main results, illustrate that the above boundedness conditions can be released. In this statement we investigate the qualitative behavior of the solution of Eq. (1.12) under the initial condition

$$x(t) = \varphi(t), \qquad -\tau \le t \le 0, \tag{1.18}$$

where $\varphi \in C_+ := \{\psi \in C([-\tau, 0], \mathbb{R}_+) \; : \; \psi(0) > 0\}$. The unique solution of Eq. (1.12) and (1.18) is denoted by $x(\varphi)(t)$. We will assume

$$\alpha_k, \tau_k \in C(\mathbb{R}_+, \mathbb{R}_+), \; (k = 1, \ldots, n), \qquad \tau := \max_{1 \le k \le n} \sup_{t \ge 0} \tau_k(t) < \infty, \tag{1.19}$$

$$\beta \in C(\mathbb{R}_+, (0, \infty)), \qquad \int_0^\infty \beta(t)\, dt = \infty, \tag{1.20}$$

and

$$0 < \underline{m} := \liminf_{t \to \infty} \frac{1}{\beta(t)} \sum_{k=1}^{n} \alpha_k(t) \qquad \text{and} \qquad \overline{m} := \limsup_{t \to \infty} \frac{1}{\beta(t)} \sum_{k=1}^{n} \alpha_k(t) < \infty. \tag{1.21}$$

We note that Eq. (1.12) has no constant positive steady-state if the function $\frac{1}{\beta(t)} \sum_{k=1}^{n} \alpha_k(t)$ is not constant. Part (i) of the next result implies that if the limit $m := \lim_{t \to \infty} \frac{1}{\beta(t)} \sum_{k=1}^{n} \alpha_k(t)$ exists, then all positive solutions converge to m, i.e., m is an asymptotic equilibrium of Eq. (1.12). But if $\underline{m} < \overline{m}$, then the limit of the solution may not exist (see, e.g., Example 3.5, where the equation has a positive periodic solution). In this case we give estimates for the limit inferior and limit superior of all positive solutions, i.e., we show that (1.12) is uniformly permanent (see Definition 2.2 below). In part (ii) and (iii) of the next theorem we give conditions implying that all positive solutions are asymptotically equivalent in the sense that the difference of any positive solutions tend to 0, even in the case when they do not converge to a constant limit.

Theorem 1.1. *Assume (1.19), (1.20) and (1.21). Then*

(i) for any $\varphi \in C_+$ the solution $x(\varphi)(t)$ of the IVP (1.12) and (1.18) obeys

$$\underline{m} \leq \liminf_{t\to\infty} x(\varphi)(t) \leq \limsup_{t\to\infty} x(\varphi)(t) \leq \overline{m}. \tag{1.22}$$

(ii) If, in addition, we assume

$$\overline{m} < 2\underline{m}, \tag{1.23}$$

then for any $\varphi, \psi \in C_+$, the solutions $x(\varphi)$ and $x(\psi)$ of the IVP (1.12) and (1.18) satisfy

$$\lim_{t\to\infty} \Big(x(\varphi)(t) - x(\psi)(t) \Big) = 0. \tag{1.24}$$

(iii) Moreover, if in addition to the assumptions of part (ii), we assume (1.12) has a positive periodic solution, say $Q(t)$, $t \geq -\tau$, then $Q(t)$ attracts all positive solutions of Eq. (1.12), i.e., (1.24) holds with $x(\psi)(t) = Q(t)$.

Our proof is based on using some relevant well-known theorem for differential inequalities of ordinary differential equations, moreover we can apply our method for differential equations with distributed delay, e.g., of the form (1.16), where techniques of [2] and [6] do not work.

Statement (ii) of Theorem 1.1 (see also Corollary 3.2 below for a more general case) is a novel result, which is interesting on its own right. One reason for this is that most of the attractivity results in the literature focus on the case when the investigated equation has a saturated equilibrium. See, e.g., [12] Section 4.8 for related results. Although, condition (1.23) is not sharp, Eq. (1.12) is simple, statement (ii) (and also (iii)) of Theorem 1.1 may initiate further research in more general equations without constant steady state solutions.

The structure of our paper is the following. In Section 2 we formulate our main results. Lemma 2.4 below shows that, under some conditions, Eq. (1.11) is persistent, i.e., all solutions are separated from 0 by a positive constant. Theorem 2.5 below gives estimations for the limit inferior and limit superior of positive solutions of Eq. (1.11). In Section 3 we show several examples including equations (1.12), (1.15), (1.14) and (1.16), where Theorem 2.5 implies that the equation is uniformly permanent, i.e., (1.22) holds for the positive solutions, and we give explicit estimations of $\underline{m}$ and $\overline{m}$. We also investigate the asymptotic equivalency of solutions of Eq. (1.13). In Section 4 we present the proofs of our main results, and in Section 5 we summarize our conclusions.

2. Main results

Throughout the manuscript we use the following notations. Let $\tau > 0$ be fixed, and $C := C([-\tau, 0], \mathbb{R})$, and also we define $\mathbb{R}_+ := [0, \infty)$. As usual, x_t denotes the function in C defined by $x_t(\theta) = x(t + \theta)$, $-\tau \leq \theta \leq 0$. Let C_+ be the space of continuous functions $\psi : [-\tau, 0] \to \mathbb{R}_+$ with $\psi(0) > 0$. We will use the notations

$$\underline{x}(\infty) := \liminf_{t\to\infty} x(t) \qquad \text{and} \qquad \overline{x}(\infty) := \limsup_{t\to\infty} x(t).$$

We consider the scalar nonlinear delay equation

$$\dot{x}(t) = r(t)\Big(g(t, x_t) - h(x(t))\Big), \qquad t \geq 0, \tag{2.1}$$

with the initial condition

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0. \tag{2.2}$$

Following [1], [2], [6] and [7] we define the following notions.

Definition 2.1. *Eq. (2.1) is said to be persistent in C_+ if any solution $x(t)$ with initial condition $\varphi \in C_+$ is bounded away from zero, i.e.,* $\liminf_{t\to\infty} x(t) > 0$.

Definition 2.2. *Eq. (2.1) is called uniformly permanent if there exist two positive numbers m and M with $m < M$ such that, for all solutions $x(t)$ of Eq. (2.1) with the initial condition $\varphi \in C_+$ satisfy*

$$m \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq M.$$

Next we list the following conditions, which will be used only whenever this is explicitly indicated:

(**A**) $r \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $r(t) > 0$ for $t > 0$ and $\int_0^\infty r(s)ds = \infty$, $g \in C(\mathbb{R}_+ \times C, \mathbb{R})$ with $g(t, \psi) \geq 0$ for $t \geq 0$ and $\psi(s) \geq 0$, $-\tau \leq s \leq 0$.

(**B**) $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $0 = h(0) < h(x_1) < h(x_2)$ for $0 < x_1 < x_2$, and for any nonnegative constants v and L satisfying $L \neq v$ the condition $\int_L^v \frac{ds}{h(v)-h(s)} = +\infty$ holds.

(**C_1**) There exists $q_1 \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ such that for any $T \geq 0$, $u > 0$ we have

$$g(t, \psi) \geq q_1(T, u), \quad \text{if} \quad t \geq T \text{ and } \psi \in C \text{ with } \psi(s) \geq u, \ -\tau \leq s \leq 0,$$

and there exist constants $T_1 \geq \tau$ and $u_1 > 0$ such that

$$q_1(T_1, u) > h(u), \qquad u \in (0, u_1].$$

(**C_2**) There exists $q_2 \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ such that for any $T \geq 0$, $u > 0$ we have

$$g(t, \psi) \leq q_2(T, u), \quad \text{if} \quad t \geq T \text{ and } \psi \in C \text{ with } \psi(s) \leq u, \ -\tau \leq s \leq 0,$$

and there exist constants $T_2 \geq \tau$ and $u_2 > 0$ such that

$$q_2(T_2, u) < h(u), \qquad u \geq u_2.$$

(**D_1**) There exists $q_1^* \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that for any $v \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\lim_{T\to\infty} v(T) = w$ we have

$$\liminf_{T\to\infty} q_1(T, v(T)) \geq q_1^*(w).$$

($\mathbf{D_2}$) There exists $q_2^* \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that for any $v \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\lim_{T\to\infty} v(T) = w$ we have

$$\limsup_{T\to\infty} q_2(T, v(T)) \le q_2^*(w).$$

We remark that from the assumed continuity of the functions r, g, h and φ, the IVP (2.1) and (2.2) has a solution, but it is not necessary unique. Any fixed solution of (2.1) corresponding to the initial function φ will be denoted by $x(\varphi)(t)$, and we assume that this solution exists on $[0, \infty)$. We also note that if h is locally Lipschitz continuous, then the integral condition in (**B**) holds.

Now we formulate our main results. The proofs of these statements will be postponed to Section 4.

The first lemma shows that all solutions of (2.1) corresponding to the initial condition $\varphi \in C_+$ are positive on $[0, \infty)$.

Lemma 2.3. *Assume that conditions (**A**) and (**B**) are satisfied. Then, for any $\varphi \in C_+$, we have that $x(\varphi)(t) > 0$ for $t \in [0, \infty)$.*

The next result implies that, under our conditions, Eq. (2.1) is persistent.

Lemma 2.4. *Let conditions (**A**) and (**B**) be satisfied. Then, for any $\varphi \in C_+$, we have*

(i) if ($\boldsymbol{C_1}$) is satisfied, then any solution $x(\varphi)(t)$ of the IVP (2.1) and (2.2) satisfies

$$\inf_{t\ge 0} x(\varphi)(t) > 0; \tag{2.3}$$

(ii) if ($\boldsymbol{C_2}$) is satisfied, then any solution $x(\varphi)(t)$ of the IVP (2.1) and (2.2) satisfies

$$\sup_{t\ge 0} x(\varphi)(t) < \infty. \tag{2.4}$$

Now we state our main result, which can be used to estimate $\liminf_{t\to\infty} x(t)$ and $\limsup_{t\to\infty} x(t)$. In the next section we will show that in many particular situations these estimations imply that Eq. (2.1) is uniformly permanent.

Theorem 2.5. *Assume (**A**) and (**B**) are satisfied. Then for any $\varphi \in C_+$, we have*

(i) if ($\boldsymbol{C_1}$) and ($\boldsymbol{D_1}$) are satisfied, then any solution $x(t) = x(\varphi)(t)$ of the IVP (2.1) and (2.2) is bounded from below on $[0, \infty)$, and

$$h^{-1}(q_1^*(\underline{x}(\infty))) \le \underline{x}(\infty); \tag{2.5}$$

(ii) if ($\boldsymbol{C_2}$) and ($\boldsymbol{D_2}$) are satisfied, then any solution $x(t) = x(\varphi)(t)$ of the IVP (2.1) and (2.2) is bounded from above on $[0, \infty)$ and

$$\overline{x}(\infty) \le h^{-1}(q_2^*(\overline{x}(\infty))). \tag{2.6}$$

Our main theorem implies the following corollary, which formulates sufficient conditions for that all positive solutions converge to a constant limit.

Corollary 2.6. *Assume all conditions* $(\boldsymbol{A})$ – $(\boldsymbol{D_2})$ *hold, moreover* $q^*(w) := q_1^*(w) = q_2^*(w)$ *for* $w \in \mathbb{R}_+$, *and there exists* $u^* > 0$ *such that*

$$q^*(u) > h(u) \quad \text{for } u \in (0, u^*) \qquad \text{and} \qquad q^*(u) < h(u) \quad \text{for } u > u^*. \tag{2.7}$$

Then, for any $\varphi \in C_+$, *any solution* $x(t) = x(\varphi)(t)$ *of the IVP (2.1) and (2.2) satisfies*

$$\lim_{t\to\infty} x(t) = u^*. \tag{2.8}$$

3. Corollaries and examples

In this section, we provide several corollaries and examples to our main results. First, we consider the equation

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t) x^p(t - \sigma_k(t)) - \beta(t) x^q(t), \qquad t \geq 0, \tag{3.1}$$

with

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0. \tag{3.2}$$

Corollary 3.1. *Consider the IVP (3.1) and (3.2), where* $0 < p < q$, $q \geq 1$,

$$0 \leq \sigma_k(t) \leq \tau, \qquad t \geq 0 \quad \text{and} \quad k = 1, \ldots, n \tag{3.3}$$

with some positive constant τ, *and* $\alpha_k, \beta \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with*

$$\beta(t) > 0 \text{ for } t > 0, \quad \int_0^\infty \beta(t)dt = \infty, \quad \lim_{t\to 0+} \frac{\alpha_k(t)}{\beta(t)} < \infty \text{ exists for } k = 1, \ldots, n, \tag{3.4}$$

and

$$\underline{m} := \liminf_{t\to\infty} \frac{\sum_{k=1}^n \alpha_k(t)}{\beta(t)} > 0 \qquad \text{and} \qquad \overline{m} := \limsup_{t\to\infty} \frac{\sum_{k=1}^n \alpha_k(t)}{\beta(t)} < \infty. \tag{3.5}$$

Then, for any initial function $\varphi \in C_+$, *any solution* $x(t) = x(\varphi)(t)$ *of the IVP (3.1) and (3.2) satisfies*

$$\underline{m}^{\frac{1}{q-p}} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \overline{m}^{\frac{1}{q-p}}. \tag{3.6}$$

Proof The proof is obtained directly from Theorem 2.5, where we can rewrite (3.1) as follows

$$\dot{x}(t) = \beta(t) \left[\sum_{k=1}^{n} \frac{\alpha_k(t)}{\beta(t)} x^p(t - \sigma_k(t)) - x^q(t) \right], \qquad t > 0. \tag{3.7}$$

Note that (3.4) yields that if $\beta(0) = 0$, then the functions $\frac{\alpha_k(t)}{\beta(t)}$ can be extended continuously to $t = 0$. For simplicity, this extended function is denoted by $\frac{\alpha_k(t)}{\beta(t)}$, as well. We can see from (3.7) that Eq. (3.1) can be written in the form (2.1) with $r(t) := \beta(t)$, $g(t, \psi) := \sum_{k=1}^{n} \frac{\alpha_k(t)}{\beta(t)} \psi^p(-\sigma_k(t))$ and $h(x) := x^q$. Since $q \geq 1$, $h(x)$ is locally Lipschitz continuous, and so conditions (**A**) and (**B**) are satisfied. Now we check that conditions ($\mathbf{C_1}$)–($\mathbf{D_2}$) are satisfied. Suppose that $\psi(s) \geq u > 0$ for $-\tau \leq s \leq 0$, then $g(t, \psi) \geq q_1(T, u)$ for $t \geq T \geq 0$, where

$$q_1(T, u) := m_T u^p, \qquad m_T := \inf_{t \geq T} \frac{\sum_{k=1}^{n} \alpha_k(t)}{\beta(t)}.$$

Therefore ($\mathbf{C_1}$) is satisfied if $m_{T_1} u^p > u^q$, or equivalently $m_{T_1} > u^{q-p}$ for some $T_1 \geq \tau$ and small positive u. Since (3.5) yields $\underline{m} = \liminf_{T \to \infty} m_T > 0$, there exist $T_1 > 0$ and $u_1 > 0$ such that

$$m_{T_1} > u_1^{q-p} \geq u^{q-p} \quad \text{for } u \in (0, u_1],$$

and hence ($\mathbf{C_1}$) is satisfied. In a similar way we can show that ($\mathbf{C_2}$) is satisfied.

To check ($\mathbf{D_1}$), suppose $v(T) \to w$ as $T \to \infty$. Then

$$\lim_{T \to \infty} q_1(T, v(T)) = \lim_{T \to \infty} m_T v^p(T) = \underline{m} w^p,$$

so ($\mathbf{D_1}$) is satisfied with $q_1^*(w) := \underline{m} w^p$. In a similar way we can check ($\mathbf{D_2}$). Thus Theorem 2.5 is applicable, so we see that

$$h^{-1}(q_1^*(\underline{x}(\infty))) \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq h^{-1}(q_2^*(\overline{x}(\infty))).$$

Hence

$$(\underline{m}\underline{x}^p(\infty))^{1/q} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq (\overline{m}\overline{x}^p(\infty))^{1/q},$$

therefore we get (3.6). □

The next result gives sufficient conditions which yield that all positive solutions are asymptotically equivalent.

Corollary 3.2. *Consider the IVP (3.1) and (3.2), where σ_k satisfy (3.3), and α_k and β satisfy (3.4) and (3.5), and suppose $1 \leq p < q$ are integers, and*

$$0 < p \cdot \overline{m}^{\frac{q-1}{q-p}} < q \cdot \underline{m}^{\frac{q-1}{q-p}}, \tag{3.8}$$

where $\underline{m}$ and $\overline{m}$ are defined in (3.5). Then, for any initial functions $\varphi, \psi \in C_+$, any corresponding solutions $x(\varphi)(t)$ and $x(\psi)(t)$ of the IVP (3.12) and (3.13) satisfy

$$\lim_{t \to \infty} \Big(x(\varphi)(t) - x(\psi)(t) \Big) = 0. \tag{3.9}$$

Proof Introduce the short notations $x_1(t) := x(\varphi)(t)$ and $x_2(t) := x(\psi)(t)$. It follows from Corollary 3.1 that

$$\underline{m}^{\frac{1}{q-p}} \leq \liminf_{t \to \infty} x_i(t) \leq \limsup_{t \to \infty} x_i(t) \leq \overline{m}^{\frac{1}{q-p}}, \qquad i = 1, 2. \tag{3.10}$$

Eq. (3.1) yields for $t \geq 0$ that

$$\dot{x}_1(t) - \dot{x}_2(t) = \sum_{k=1}^{n} \alpha_k(t)\Big(x_1^p(t-\sigma_k(t)) - x_2^p(t-\sigma_k(t))\Big) - \beta(t)\Big(x_1^q(t) - x_2^q(t)\Big).$$

Therefore the function $w(t) := x_1(t) - x_2(t)$ satisfies

$$\dot{w}(t) = \sum_{k=1}^{n} \alpha_k(t)a_k(t)w(t-\sigma_k(t)) - \beta(t)b(t)w(t), \qquad t \geq 0, \tag{3.11}$$

where

$$a_k(t) := \sum_{\ell=0}^{p-1} x_1^\ell(t-\sigma_k(t))x_2^{p-1-\ell}(t-\sigma_k(t)), \quad k = 1, \dots, n$$

and

$$b(t) := \sum_{\ell=0}^{q-1} x_1^\ell(t)x_2^{q-1-\ell}(t).$$

The definitions of $a_k(t)$, $b(t)$, relation (3.10) and assumption (3.8) imply

$$\limsup_{t\to\infty} \frac{\sum_{k=1}^n \alpha_k(t)a_k(t)}{\beta(t)b(t)} \leq \frac{p \cdot \overline{m}^{\frac{p-1}{q-p}}}{q \cdot \underline{m}^{\frac{q-1}{q-p}}} \limsup_{t\to\infty} \frac{\sum_{k=1}^n \alpha_k(t)}{\beta(t)} = \frac{p \cdot \overline{m}^{\frac{p-1}{q-p}+1}}{q \cdot \underline{m}^{\frac{q-1}{q-p}}} < 1.$$

Then a simple generalization of Theorem 3.1 of [9] yields that the trivial solution of Eq. (3.11) is globally asymptotically stable, so $\lim_{t\to\infty} w(t) = 0$, which completes the proof of the statement. □

Next, we consider the special case of (3.1), which is identical to Eq. (1.12)

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t)x(t-\sigma_k(t)) - \beta(t)x^2(t), \qquad t \geq 0, \tag{3.12}$$

with

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0. \tag{3.13}$$

Corollary 3.1 immediately implies the estimate obtained in [2], but under weaker conditions, since the boundedness conditions (1.17) of the coefficients are not required.

Corollary 3.3. *Consider the IVP (3.12) and (3.13), where σ_k satisfy (3.3), and α_k and β satisfy (3.4) and (3.5). Then,*

(i) for any initial function $\varphi \in C_+$, the unique solution $x(t) = x(\varphi)(t)$ of the IVP (3.12) and (3.13) satisfies

$$\underline{m} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \overline{m}, \tag{3.14}$$

where $\underline{m}$ and $\overline{m}$ are defined in (3.5).

(ii) Moreover, if in addition

$$\overline{m} < 2\underline{m}, \tag{3.15}$$

then any positive solutions of Eq. (3.12) are asymptotically equivalent, i.e., (3.9) holds.

Example 3.4. Consider the differential equation

$$\dot{x}(t) = t(2+\cos t)x(t-\tau) - tx^2(t), \qquad t \geq 0. \tag{3.16}$$

It is clear that (3.16) is a special case of (3.12) with $n = 1$, $\alpha_1(t) = t(2+\cos t)$, $\beta(t) = t$, and relations (3.4) and (3.5) hold. We get

$$\underline{m} = \liminf_{t\to\infty} \frac{\alpha_1(t)}{\beta(t)} = \liminf_{t\to\infty}(2+\cos t) = 1,$$

$$\overline{m} = \limsup_{t\to\infty} \frac{\alpha_1(t)}{\beta(t)} = \limsup_{t\to\infty}(2+\cos t) = 3.$$

Hence Corollary 3.3 yields that all solutions of (3.16) corresponding to an initial function $\varphi \in C_+$ satisfy

$$1 \leq \underline{x}(\varphi)(\infty) \leq \overline{x}(\varphi)(\infty) \leq 3.$$

We note that the results of [2] and [6] cannot be applied for (3.16), since the coefficients do not satisfy (1.17).

In Figure 1 we plotted the solutions of Eq. 3.16 starting from the constant initial functions $\varphi(t) = 0.2$, $\varphi(t) = 1$ and $\varphi(t) = 2$. We can see from the figure (and from other numerical runnings) that the above estimates hold, moreover, all solutions seem to be asymptotically equivalent, despite of that condition (3.15) does not hold in this example. □

The next example shows that estimate (3.14) is sharp in some cases.

Example 3.5. For $\tau > \pi$ consider the differential equation

$$\dot{x}(t) = \Big(\frac{\pi}{\tau}\sin\frac{4\pi}{\tau}t + e^{\frac{1}{2}\sin^2\frac{2\pi}{\tau}t}\Big)x(t-\tau) - x^2(t), \qquad t \geq 0. \tag{3.17}$$

An application of Corollary 3.3 gives that the solutions of (3.17) corresponding to an initial function $\varphi \in C_+$ satisfy

$$\underline{m}_\tau \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \overline{m}_\tau,$$

where

$$\underline{m}_\tau := \liminf_{t\to\infty}\left(\frac{\pi}{\tau}\sin\frac{4\pi}{\tau}t + e^{\frac{1}{2}\sin^2\frac{2\pi}{\tau}t}\right)$$

and

$$\overline{m}_\tau := \limsup_{t\to\infty}\left(\frac{\pi}{\tau}\sin\frac{4\pi}{\tau}t + e^{\frac{1}{2}\sin^2\frac{2\pi}{\tau}t}\right).$$

Simple calculation shows that the function

$$x(t) = e^{\frac{1}{2}\sin^2 \frac{2\pi}{\tau} t}, \qquad t \geq 0$$

is a positive solution of Eq. (3.17), and $\underline{x}(\infty) = 1$, $\overline{x}(\infty) = \sqrt{e}$. Therefore for $\tau > \pi$

$$\underline{m}_\tau \leq 1 < \sqrt{e} \leq \overline{m}_\tau.$$

It is easy to see that $\underline{m}_\tau \to 1$ and $\overline{m}_\tau \to \sqrt{e}$ as $\tau \to \infty$, so our estimations are getting sharper and sharper as $\tau \to \infty$.

We note that condition (3.15) holds for large enough τ, so then Corollary 3.2 yields immediately that for such τ the function $e^{\frac{1}{2}\sin^2 \frac{2\pi}{\tau} t}$ is the only positive periodic solution of (3.17), and it attracts all positive solutions. □

Next we consider a scalar delay differential equation with more general nonlinearity

$$\dot{x}(t) = \alpha(t) f(x(t - \sigma(t))) - \beta(t) h(x(t)), \qquad t \geq 0, \tag{3.18}$$

with

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0. \tag{3.19}$$

Corollary 3.6. *Consider the IVP (3.18) and (3.19), where the delay function σ satisfies $0 \leq \sigma(t) \leq \tau$ for $t \geq 0$ with some positive constants τ, and $\alpha, \beta \in C(\mathbb{R}_+, \mathbb{R}_+)$ with*

$$\beta(t) > 0 \quad \text{for } t > 0, \qquad \int_0^\infty \beta(t)dt = \infty, \qquad 0 \leq \lim_{t \to 0+} \frac{\alpha(t)}{\beta(t)} < \infty \text{ exists}, \tag{3.20}$$

and

$$\underline{m} := \liminf_{t \to \infty} \frac{\alpha(t)}{\beta(t)} > 0 \qquad \text{and} \qquad \overline{m} := \limsup_{t \to \infty} \frac{\alpha(t)}{\beta(t)} < \infty, \tag{3.21}$$

$f, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ are increasing functions with $h(0) = 0$, h is locally Lipschitz continuous, and

$$G(u) := \frac{h(u)}{f(u)} \text{ is monotone increasing}, \quad \lim_{u \to 0} G(u) = 0 \quad \text{and} \quad \lim_{u \to \infty} G(u) = \infty. \tag{3.22}$$

Then, for any initial function $\varphi \in C_+$, any solution $x(t) = x(\varphi)(t)$ of the IVP (3.18) and (3.19) satisfies

$$G^{-1}(\underline{m}) \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq G^{-1}(\overline{m}). \tag{3.23}$$

Proof We rewrite (3.18) as

$$\dot{x}(t) = \beta(t) \left[\frac{\alpha(t)}{\beta(t)} f(x(t - \sigma(t))) - h(x(t)) \right], \qquad t \geq 0. \tag{3.24}$$

We can see from (3.24) that $r(t) := \beta(t)$ and $g(t, \psi) := \frac{\alpha(t)}{\beta(t)} f(\psi(-\sigma(t)))$. It is clear that conditions (**A**) and (**B**) hold. We check that conditions ($\mathbf{C_1}$)–($\mathbf{D_2}$)

are satisfied. Suppose that $\psi(s) \ge u > 0$ for $-\tau \le s \le 0$, then $g(t, x_t) \ge q_1(T, u)$ for $t \ge T$, where

$$q_1(T, u) := m_T f(u), \qquad m_T := \inf_{t \ge T} \frac{\alpha(t)}{\beta(t)}.$$

Hence $(\mathbf{C_1})$ is satisfied if $m_{T_1} f(u) > h(u)$, or equivalently

$$m_{T_1} > G(u) \tag{3.25}$$

for some $T_1 \ge \tau$ and for small enough positive u. It follows from (3.21) that there exists $T_1 > 0$ such that $m_{T_1} > 0$. Using $\lim_{u \to 0} G(u) = 0$, there exists $u_1 > 0$ such that $0 < G(u_1) < m_{T_1}$. Thus we have that (3.25) holds for $u \in (0, u_1]$, and hence $(\mathbf{C_1})$ is satisfied. Similarly, we can check $(\mathbf{C_2})$.

To show $(\mathbf{D_1})$, suppose that $\lim_{T \to \infty} v(T) = w$, and consider

$$\lim_{T \to \infty} q_1(T, v(T)) = \lim_{T \to \infty} m_T f(v(T)) = \underline{m} f(w),$$

so $(\mathbf{D_1})$ is satisfied with $q_1^*(w) := \underline{m} f(w)$. In a similar way we can check $(\mathbf{D_2})$. Thus Theorem 2.5 is applicable, so we see that

$$h^{-1}(q_1^*(\underline{x}(\infty))) \le \underline{x}(\infty) \le \overline{x}(\infty) \le h^{-1}(q_2^*(\overline{x}(\infty))).$$

Hence

$$\underline{m} f(\underline{x}(\infty)) \le h(\underline{x}(\infty)) \le h(\overline{x}(\infty)) \le \overline{m} f(\overline{x}(\infty)),$$

and therefore, using (3.22), we get (3.23). □

Corollary 3.7. *Suppose all conditions of Corollary 3.6 hold, moreover*

$$0 < m := \lim_{t \to \infty} \frac{\alpha(t)}{\beta(t)} < \infty \tag{3.26}$$

exists, and there exists $u^* > 0$ such that

$$m f(u) > h(u) \quad \text{for } u \in (0, u^*) \qquad \text{and} \qquad m f(u) < h(u) \quad \text{for } u > u^*. \tag{3.27}$$

Then, for any initial function $\varphi \in C_+$, any solution $x(t) = x(\varphi)(t)$ of the IVP (3.18) and (3.19) satisfies

$$\lim_{t \to \infty} x(t) = u^*. \tag{3.28}$$

Proof It follows from the proof of Corollary 3.6 that $q_1^*(w) = q_2^*(w) = m f(w)$, $w \in \mathbb{R}_+$, so Corollary 2.6 yields (3.28). □

Example 3.8. *Consider*

$$\dot{x}(t) = t\Big((2 + \frac{1}{t+1})x(t - 3 - \sin t) - x^2(t)\Big), \qquad t \ge 0. \tag{3.29}$$

Then all conditions of Corollary 3.7 hold with $m = 2$ and $u^* = 2$, so the solutions of (3.29) corresponding to an initial function $\varphi \in C_+$ satisfies

$$\lim_{t \to \infty} x(t) = 2.$$

Now we consider the IVP

$$\dot{x}(t) = \alpha(t) \int_{-\tau}^{0} f(s, x(t+s))\, ds - \beta(t) g(x(t)), \qquad t \geq 0 \tag{3.30}$$

with the initial condition

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0. \tag{3.31}$$

Corollary 3.9. *Consider the IVP (3.30) and (3.31), where $\alpha, \beta \in C(\mathbb{R}_+, \mathbb{R}_+)$ obey (3.20) and (3.21), $f \in C([-\tau, 0] \times \mathbb{R}, \mathbb{R}_+)$ is increasing in its second variable, $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing function with $h(0) = 0$, h is locally Lipschitz continuous, and*

$$G(u) := \frac{h(u)}{\int_{-\tau}^{0} f(s,u)\, ds} \text{ is monotone increasing, } \lim_{u \to 0} G(u) = 0, \ \lim_{u \to \infty} G(u) = \infty.$$

Then, for any initial function $\varphi \in C_+$, any solution $x(t) = x(\varphi)(t)$ of the IVP (3.30) and (3.31) satisfies

$$G^{-1}(\underline{m}) \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq G^{-1}(\overline{m}). \tag{3.32}$$

Proof The proof is similar to that of Corollary 3.6, so it is omitted. □

Next we consider the IVP

$$\dot{x}(t) = \frac{\alpha(t) x(t - \sigma(t))}{1 + \gamma(t) x(t - \sigma(t))} - \beta(t) x^2(t), \qquad t \geq 0, \tag{3.33}$$

with the initial condition

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0. \tag{3.34}$$

We show that, under weak conditions on the coefficients, Theorem 2.5 is applicable to estimate $\underline{x}(\infty)$ and $\overline{x}(\infty)$.

Corollary 3.10. *Suppose $0 \leq \sigma(t) \leq \tau$ with some $\tau > 0$, the coefficients $\alpha, \beta, \gamma \in C(\mathbb{R}_+, \mathbb{R}_+)$ with*

$$\beta(t) > 0, \ t > 0, \quad \int_0^\infty \beta(t) dt = \infty, \quad \lim_{t \to 0+} \frac{\alpha(t)}{\beta(t)} < \infty \text{ exists}, \quad 0 < \liminf_{t \to \infty} \gamma(t), \tag{3.35}$$

and for some $\varepsilon > 0$

$$\underline{m}_\varepsilon := \liminf_{t \to \infty} \frac{-1 + \sqrt{1 + \frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)} > 0 \tag{3.36}$$

and

$$\overline{m}_\varepsilon := \limsup_{t \to \infty} \frac{-1 + \sqrt{1 + \frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)} < \infty. \tag{3.37}$$

Furthermore, suppose there exist functions q_1^* and q_2^* so that if $\lim_{T \to \infty} v(T) = w$, then

$$\liminf_{T \to \infty} \inf_{t \geq T} \frac{\frac{\alpha(t)}{\beta(t)} v(T)}{1 + \gamma(t) v(T)} \geq q_1^*(w) \tag{3.38}$$

and

$$\limsup_{T\to\infty}\sup_{t\geq T}\frac{\frac{\alpha(t)}{\beta(t)}v(T)}{1+\gamma(t)v(T)}\leq q_2^*(w). \tag{3.39}$$

Then, for any initial function $\varphi\in C_+$, the solution $x(t)=x(\varphi)(t)$ of the IVP (3.33) and (3.34) satisfies

$$\sqrt{q_1^*(\underline{x}(\infty))}\leq\underline{x}(\infty)\leq\overline{x}(\infty)\leq\sqrt{q_2^*(\overline{x}(\infty))}. \tag{3.40}$$

Proof We can rewrite (3.33) as follows

$$\dot{x}(t)=\beta(t)\left[\frac{\frac{\alpha(t)}{\beta(t)}x(t-\sigma(t))}{1+\gamma(t)x(t-\sigma(t))}-x^2(t)\right],\qquad t\geq 0,$$

where $\frac{\alpha(t)}{\beta(t)}$ denotes the continuous extension of the function to $t=0$ if $\beta(0)=0$. Let us define $r(t):=\beta(t)$, $g(t,\psi):=\frac{\frac{\alpha(t)}{\beta(t)}\psi(-\sigma(t))}{1+\gamma(t)\psi(-\sigma(t))}$ and $h(x):=x^2$. It is clear that conditions (**A**) and (**B**) are satisfied. We check that conditions ($\mathbf{C_1}$)–($\mathbf{D_2}$) are satisfied. Suppose that $\psi(s)\geq u>0$ for $-\tau\leq s\leq 0$, then $g(t,x_t)\geq q_1(T,u)$ for $t\geq T$, where

$$q_1(T,u):=\inf_{t\geq T}\frac{\frac{\alpha(t)}{\beta(t)}u}{1+\gamma(t)u}.$$

Thus ($\mathbf{C_1}$) is satisfied if for some $T_1\geq\tau$, $\varepsilon>0$ and small enough $u>0$

$$\frac{\frac{\alpha(t)}{\beta(t)}u}{1+\gamma(t)u}\geq(1+\varepsilon)u^2,\qquad t\geq T_1,$$

or equivalently

$$(1+\varepsilon)\gamma(t)u^2+(1+\varepsilon)u-\frac{\alpha(t)}{\beta(t)}\leq 0,\qquad t\geq T_1. \tag{3.41}$$

Relation (3.35) implies there exists $T_1\geq\tau$ and $\varepsilon>0$ such that $\gamma(t)>0$ for $t\geq T_1$ and

$$u_1:=\inf_{t\geq T_1}\frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)}>0. \tag{3.42}$$

So for $t\geq T_1$

$$(1+\varepsilon)\gamma(t)y^2+(1+\varepsilon)y-\frac{\alpha(t)}{\beta(t)}=0$$

is a quadratic equation, and (3.36) yields that it has a negative solution and a positive solution

$$\frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)}.$$

Therefore (3.42) yields (3.41), and hence $q_1(T_1,u)>u^2$ holds for $0<u\leq u_1$. In a similar way we can show that ($\mathbf{C_2}$) is satisfied.

Assumption ($\mathbf{D_1}$) follows from (3.38), since

$$\liminf_{T\to\infty} q_1(T, v(T)) = \liminf_{T\to\infty} \inf_{t\geq T} \frac{\frac{\alpha(t)}{\beta(t)} v(T)}{1+\gamma(t)v(T)}.$$

Assumption ($\mathbf{D_2}$) can be shown similarly. Then Theorem 2.5 yields (3.40). □

The next two corollaries illustrate two cases when relations (3.38) and (3.39) can be checked easily. First consider the case when $\gamma(t) \to \infty$ as $t \to \infty$.

Corollary 3.11. *Suppose $0 \leq \sigma(t) \leq \tau$ with some $\tau > 0$, the coefficients $\alpha, \beta, \gamma \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy (3.35), (3.36) and (3.37). Furthermore, suppose*

$$\lim_{t\to\infty} \gamma(t) = \infty. \tag{3.43}$$

Then, for any initial function $\varphi \in C_+$, the solution $x(t) = x(\varphi)(t)$ of the IVP (3.33) and (3.34) satisfies

$$\underline{m} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \overline{m}, \tag{3.44}$$

where $\underline{m} := \underline{m}_0$ and $\overline{m} := \overline{m}_0$ are defined in (3.36) and (3.37) with $\varepsilon = 0$.

Proof Assumption (3.43) yields

$$\begin{aligned}
\underline{m}_\varepsilon &= \liminf_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)} \\
&= \liminf_{t\to\infty} \Big(-\frac{1}{2\gamma(t)} + \sqrt{\frac{1}{4\gamma^2(t)} + \frac{\alpha(t)}{(1+\varepsilon)\beta(t)\gamma(t)}} \Big) \\
&= \liminf_{t\to\infty} \sqrt{\frac{\alpha(t)}{(1+\varepsilon)\beta(t)\gamma(t)}} \\
&= \frac{1}{\sqrt{1+\varepsilon}} \underline{m},
\end{aligned}$$

and similarly,

$$\overline{m}_\varepsilon = \limsup_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)} = \limsup_{t\to\infty} \sqrt{\frac{\alpha(t)}{(1+\varepsilon)\beta(t)\gamma(t)}} = \frac{1}{\sqrt{1+\varepsilon}} \overline{m}.$$

To check (3.38), suppose $\lim_{T\to\infty} v(T) = w$, and let $\varepsilon > 0$ be fixed. Then, for large enough t, we have $\frac{\alpha(t)}{\beta(t)\gamma(t)} \geq \underline{m}_\varepsilon^2$. Then we have

$$\begin{aligned}
\liminf_{T\to\infty} \inf_{t\geq T} \frac{\frac{\alpha(t)}{\beta(t)} v(T)}{1+\gamma(t)v(T)} &= \liminf_{T\to\infty} \inf_{t\geq T} \frac{\frac{\alpha(t)}{\beta(t)\gamma(t)} v(T)}{\frac{1}{\gamma(t)} + v(T)} \\
&\geq \liminf_{T\to\infty} \frac{\underline{m}_\varepsilon^2 v(T)}{\frac{1}{\inf_{t\geq T}\gamma(t)} + v(T)} \\
&= \underline{m}_\varepsilon^2.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows

$$\liminf_{T\to\infty} \inf_{t\geq T} \frac{\frac{\alpha(t)}{\beta(t)} v(T)}{1+\gamma(t)v(T)} \geq \underline{m}^2,$$

i.e., $q_1^*(w) = \underline{m}^2$ can be selected in $(\mathbf{D_1})$. Similar calculation shows that

$$\limsup_{T\to\infty} \sup_{t\geq T} \frac{\frac{\alpha(t)}{\beta(t)} v(T)}{1+\gamma(t)v(T)} \leq \overline{m}^2.$$

Then Theorem 2.5 yields (3.44). □

Example 3.12. Consider the equation

$$\dot{x}(t) = \frac{(1+\cos^2 t)x(t-\tau)}{1+t(\delta+\sin^2 t)x(t-\tau)} - \frac{1}{t+1}x^2(t), \qquad t \geq 0, \tag{3.45}$$

where $\delta > 0$ with the initial condition (3.34), i.e., let $\alpha(t) = 1+\cos^2 t$, $\beta(t) = \frac{1}{t+1}$ and $\gamma(t) = t(\delta + \sin^2 t)$ in (3.33). Clearly, relation (3.35) holds. To check (3.36) with $\varepsilon = 0$, we have

$$\begin{aligned}
\liminf_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{\beta(t)}}}{2\gamma(t)} &= \liminf_{t\to\infty} \Big(-\frac{1}{2\gamma(t)} + \sqrt{\frac{1}{4\gamma^2(t)} + \frac{\alpha(t)}{\beta(t)\gamma(t)}}\Big) \\
&= \liminf_{t\to\infty} \sqrt{\frac{\alpha(t)}{\beta(t)\gamma(t)}} \\
&= \liminf_{t\to\infty} \sqrt{\frac{(1+\cos^2 t)(t+1)}{t(\delta+\sin^2 t)}} \\
&\geq \sqrt{\frac{1}{\delta+1}}.
\end{aligned}$$

Similarly, (3.37) holds since

$$\limsup_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{\beta(t)}}}{2\gamma(t)} = \limsup_{t\to\infty} \sqrt{\frac{(1+\cos^2 t)(t+1)}{t(\delta+\sin^2 t)}} \leq \sqrt{\frac{2}{\delta}}.$$

Then Corollary 3.11 yields the solutions corresponding to initial function $\varphi \in C_+$ satisfy5

$$\sqrt{\frac{1}{\delta+1}} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \sqrt{\frac{2}{\delta}}.$$

5 For $\delta = 0.8$ the above estimates give $0.7454 \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq 1.5811$. In Figure 2 we display numerically generated solutions using the initial functions $\varphi(t) = 0.2$, $\varphi(t) = 0.5$ and $\varphi(t) = 2$. These runnings indicate that the solutions are asymptotically equivalent. □

In the case when $\gamma(t)$ and $\frac{\alpha(t)}{\beta(t)}$ are bounded, we can give an explicit estimates in (3.38) and (3.39), so we obtain explicit estimates of $\underline{x}(\infty)$ and $\overline{x}(\infty)$.

Corollary 3.13. *Suppose* $0 \le \sigma(t) \le \tau$ *with some* $\tau > 0$*, and the coefficients* $\alpha, \beta, \gamma \in C([0,\infty), \mathbb{R}_+)$ *satisfy (3.35). Moreover, suppose*

$$0 < \underline{m} := \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)} \le \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)} =: \overline{m} < \infty,$$

and

$$0 < \underline{l} := \liminf_{t\to\infty} \gamma(t) \le \limsup_{t\to\infty} \gamma(t) =: \bar{l} < \infty.$$

Then the solutions of the IVP (3.33) and (3.34) with $\varphi \in C_+$ *satisfy*

$$\frac{-1+\sqrt{1+4\underline{m}\bar{l}}}{2\bar{l}} \le \underline{x}(\infty) \le \overline{x}(\infty) \le \frac{-1+\sqrt{1+4\overline{m}\underline{l}}}{2\underline{l}}. \tag{3.46}$$

Proof To check (3.38) we consider

$$\lim_{T\to\infty} \inf_{t\ge T} \frac{\frac{\alpha(t)}{\beta(t)} v(T)}{1+\gamma(t)v(T)} \ge \lim_{T\to\infty} \frac{\inf_{t\ge T} \frac{\alpha(t)}{\beta(t)} v(T)}{1+\sup_{t\ge T}\gamma(t)v(T)} = \frac{\underline{m}v(T)}{1+\bar{l}v(T)},$$

so (3.38) holds with

$$q_1^*(w) = \frac{\underline{m}w}{1+\bar{l}w}.$$

Similarly, the function

$$q_2^*(w) = \frac{\overline{m}w}{1+\underline{l}w}$$

satisfies (3.39). Then (3.40) implies (3.46). □

Example 3.14. Consider the differential equation

$$\dot{x}(t) = \frac{t(3+\cos t + \frac{4}{2t+1})x(t-\tau)}{1+(2+\sin t)x(t-\tau)} - tx^2(t), \qquad t \ge 0 \tag{3.47}$$

with the initial condition (3.34). Then we see that

$$\begin{aligned} \underline{m} &:= \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)} = \liminf_{t\to\infty} \left[3+\cos t + \frac{4}{2t+1}\right] = 2, \\ \overline{m} &:= \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)} = \limsup_{t\to\infty} \left[3+\cos t + \frac{4}{2t+1}\right] = 4, \\ \underline{l} &:= \liminf_{t\to\infty} \gamma(t) = \liminf_{t\to\infty} (2+\sin t) = 1 \end{aligned}$$

and

$$\bar{l} := \limsup_{t\to\infty} \gamma(t) = \limsup_{t\to\infty} (2+\sin t) = 3.$$

Substituting in (3.46) we find that

$$0.66666\ldots = \frac{-1+\sqrt{25}}{6} \le \underline{x}(\infty) \le \overline{x}(\infty) \le \frac{-1+\sqrt{17}}{2} = 1.56155\ldots.$$

□

Finally we consider

$$\dot{x}(t) = \sum_{k=1}^{n} \frac{\alpha_k(t)x(t-\sigma_k(t))}{1+\gamma_k(t)x(t-\sigma_k(t))} - a\beta(t)x(t) - \beta(t)x^2(t), \qquad t \geq 0, \tag{3.48}$$

where $a > 0$, and we associate the initial condition

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0. \tag{3.49}$$

Note that a slightly more general version of Eq (3.48) was studied in [6] where $a\beta(t)$ was replaced by a function $\mu(t)$.

Corollary 3.15. *Suppose $a > 0$, $0 \leq \sigma_k(t) \leq \tau$ with some $\tau > 0$, and the coefficients $\alpha_k, \beta, \gamma_k \in C([0,\infty), \mathbb{R}_+)$ $(k = 1, \ldots, n)$ satisfy (3.35). Moreover, suppose*

$$0 < \underline{m}_k := \liminf_{t\to\infty} \frac{\alpha_k(t)}{\beta(t)} \leq \limsup_{t\to\infty} \frac{\alpha_k(t)}{\beta(t)} =: \overline{m}_k < \infty,$$

$$0 < \underline{l} := \min_{k=1,\ldots,n} \liminf_{t\to\infty} \gamma_k(t) \leq \max_{k=1,\ldots,n} \limsup_{t\to\infty} \gamma_k(t) =: \bar{l} < \infty$$

and

$$\sum_{k=1}^{n} \underline{m}_k > a.$$

Then the solutions of the IVP (3.48) and (3.49) with $\varphi \in C_+$ satisfy

$$\frac{-(1+a\bar{l}) + \sqrt{(1+a\bar{l})^2 - 4\bar{l}(a - \sum_{k=1}^{n} \underline{m}_k)}}{2\bar{l}} \leq \underline{x}(\infty) \tag{3.50}$$

and

$$\overline{x}(\infty) \leq \frac{-(1+a\underline{l}) + \sqrt{(1+a\underline{l})^2 - 4\underline{l}(a - \sum_{k=1}^{n} \overline{m}_k)}}{2\underline{l}}. \tag{3.51}$$

Proof The proof is similar to that of Corollary 3.13 using the function $h(u) = au + u^2$, so it is omitted here. □

Example 3.16. Consider the differential equation

$$\dot{x}(t) = \frac{(2+\sin t)x(t-\tau)}{1+x(t-\tau)} - ax(t) - x^2(t), \qquad t \geq 0 \tag{3.52}$$

with the initial condition (3.49). Here $n = 1$, $\alpha_1 = 2 + \sin t$, $\gamma_1(t) = 1$, $\beta(t) = 1$, and so $\underline{l} = \bar{l} = 1$, $\underline{m}_1 = 1$ and $\overline{m}_1 = 3$.

Consider first the case when $a = 0.1$. Then (3.50) and (3.51) yield the estimates

$$0.5 \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq 1.2114.$$

Note that Theorem 3.2 of [6] yields the estimates

$$0.45 = \frac{\sum_{k=1}^{n} \inf_{t\geq 0} \alpha_k(t) - a \sup_{t\geq 0} \beta(t)}{\sup_{t\geq 0} \beta(t) + \sum_{k=1}^{n} \inf_{t\geq 0} \alpha_k(t) \sup_{t\geq 0} \gamma_k(t)} \leq \underline{x}(\infty)$$

and

$$\overline{x}(\infty) \le \limsup_{t\to\infty} \frac{1}{\beta(t)} \sum_{k=1}^{n} \alpha_k(t) - a = 2.9,$$

so for this example our result gives better estimates.

Next consider the case when $a = 0.2$. Then our estimates (3.50) and (3.51) give

$$0.3798 \le \underline{x}(\infty) \le \overline{x}(\infty) \le 1.1204.$$

If we apply Theorem 3.2 of [6] then we get the estimates

$$0.4 \le \underline{x}(\infty) \le \overline{x}(\infty) \le 2.8,$$

where the lower estimate is better than ours, but the upper estimate is worse. □

4. Proof of the main result

In the proof of our main result, we compare the solutions of equation (2.1) with that of the associated ordinary differential equation

$$\dot{y}(t) = r(t)\Big(c - h(y(t))\Big), \qquad t \ge T \ge 0 \tag{4.1}$$

with the initial condition

$$y(T) = y^*, \tag{4.2}$$

where $c \ge 0$, and r and h satisfy (**A**) and (**B**). We will show in Lemma 4.1 below that for all $(T, y^*, c) \in (\mathbb{R}_+ \times (0,\infty) \times \mathbb{R}_+)$ the IVP (4.1) and (4.2) has a unique solution which is denoted by $y(t) = y(T, y^*, c)(t)$.

First, we prove some basic properties of the solutions of the IVP (4.1) and (4.2).

Lemma 4.1. *Let (**A**) and (**B**) be satisfied. Then for any $T \ge 0$, $y^* > 0$ and $c \ge 0$ the corresponding solution $y(T, y^*, c)(t)$ of the IVP (4.1) and (4.2) is uniquely defined on $[T,\infty)$, moreover we have*

(i) *$c > 0$ and $0 < y^* < h^{-1}(c)$ yield that*

$$0 < y(T, y^*, c)(t) < h^{-1}(c), \qquad \dot{y}(T, y^*, c)(t) > 0, \quad t \ge T$$

and

$$\lim_{t\to\infty} y(T, y^*, c)(t) = h^{-1}(c);$$

(ii) *$y^* = h^{-1}(c)$ yields that $y(T, y^*, c)(t) = h^{-1}(c), \quad t \ge T$;*

(iii) *$c \ge 0$ and $y^* > h^{-1}(c)$ yield that*

$$y(T, y^*, c)(t) > h^{-1}(c), \qquad \dot{y}(T, y^*, c)(t) < 0, \quad t \ge T$$

and

$$\lim_{t\to\infty} y(T, y^*, c)(t) = h^{-1}(c).$$

Proof It is clear from condition (**B**) that the IVP (4.1) and (4.2) has at least one solution for all $(T, y^*, c) \in (\mathbb{R}_+ \times (0,\infty) \times \mathbb{R}_+)$. Any of the corresponding solution $y(t) = y(T, y^*, c)(t)$ is considered. First we show that if $c \geq 0$ and $y^* \neq h^{-1}(c)$, then $y(t) \neq h^{-1}(c)$ for all $t \geq T$. Suppose that there exists a $t_1 > T$ such that $y(t_1) = h^{-1}(c)$. Thus, by separating variables in (4.1) and integrating from T to t_1, we get

$$\int_T^{t_1} \frac{\dot{y}(t)}{c - h(y(t))} dt = \int_T^{t_1} r(t) dt.$$

Introducing the new variable $u = y(t)$ and using (**B**) with $v = h^{-1}(c)$ we get

$$\infty = \int_{y^*}^{h^{-1}(c)} \frac{1}{c - h(u)} du = \int_T^{t_1} r(t) dt,$$

which contradicts the continuity of r. Thus $y(t) \neq h^{-1}(c)$ for $t \geq T$. Note that for $c = 0$ and $y^* > 0$, the above result yields that $y(t) > 0$ for all $t \geq T$.

Now let us prove part (i). Since $0 < y(T) < h^{-1}(c)$, then either $0 < y(t) < h^{-1}(c)$ for any $t \geq T$ and we are done, or there exists a $t_2 > T$ such that $0 < y(t) < h^{-1}(c)$ for $0 < t < t_2$ and either $y(t_2) = 0$ or $y(t_2) = h^{-1}(c)$. But this later case is not possible, since $y(t) \neq h^{-1}(c)$ for all $t \geq T$. If $y(t_2) = 0$, then one can easily see that $\dot{y}(t_2) \leq 0$. On the other hand, we get by (**A**), (**B**) and from (4.1) that

$$\dot{y}(t_2) = r(t_2)[c - h(y(t_2))] = r(t_2)[c - h(0)] = c r(t_2) > 0,$$

which is a contradiction. Hence $0 < y(t) < h^{-1}(c)$ for any $t \geq T$, and therefore $\dot{y}(t) > 0$. Since $y(t)$ is bounded, the solution $y(t)$ exists for all $t \geq T$, and since it is monotone increasing, $y(t)$ has a finite limit at ∞, and

$$N := \lim_{t \to \infty} y(t) \leq h^{-1}(c).$$

We show that $N = h^{-1}(c)$. Otherwise $N < h^{-1}(c)$, in this case since $\dot{y}(t) > 0$, by integrating (4.1) from T to t we get

$$y(t) = y(T) + \int_T^t r(s)[c - h(y(s))] ds \geq y(T) + \int_T^t r(s)[c - h(N)] ds,$$

and as $t \to \infty$ we have by (**A**) that

$$N \geq y(T) + [c - h(N)] \int_T^\infty r(s) ds = \infty.$$

This contradicts with the boundedness of $y(t)$, and hence

$$N = h^{-1}(c).$$

Now we prove part (ii). If $y(T) = h^{-1}(c)$, then it is clear that $y(t) = h^{-1}(c)$ is an equilibrium solution of (4.1) and (4.2), and it is easy to argue that $y(t) = h^{-1}(c)$ is the unique solution in this case.

The proof of part (iii) is similar to the proof of part (i), so it is omitted here.

Finally, we show the uniqueness of the solution. Let $T \geq 0$, $y^* > 0$ and $c \geq 0$ be fixed. Suppose both y_1 and y_2 satisfy the corresponding IVP (4.1) with (4.2). It follows from properties (i)–(iii) that both solutions exist on $[T, \infty)$, and $y_1(t) > 0$ and $y_2(t) > 0$ for all $t \geq T$. Suppose there exist $t_2 > T$ such that $y_1(t_2) > y_2(t_2)$ (the opposite case can be treated similarly). Then there exists $t_1 \in [T, t_2)$ such that $y_1(t_1) = y_2(t_1)$ and $y_1(t) > y_2(t)$ for $t \in (t_1, t_2)$. Define $z(t) := y_1(t) - y_2(t)$. Then z is continuously differentiable, $z(t_1) = 0$, $z(t) > 0$ for $t \in (t_1, t_2)$. On the other hand, Eq. (4.1) and the strict monotonicity of h imply

$$\dot{z}(t) = \dot{y}_1(t) - \dot{y}_2(t) = r(t)\Big(h(y_2(t)) - h(y_1(t))\Big) < 0, \quad t \in (t_1, t_2),$$

which is a contradiction. This yields that $y_1(t) = y_2(t)$ must hold for $t > T$. □

Proof of Lemma 2.3 Let $x(t) = x(\varphi)(t)$ be any solution of the IVP (2.1) and (2.2). Since $x(0) = \varphi(0) > 0$, there exists a $\delta > 0$ such that $x(t) > 0$ for $0 \leq t < \delta$. If $\delta = \infty$, then the proof is completed. Otherwise, there exists a $t_1 \in (0, \infty)$ such that $x(t) > 0$ for $0 \leq t < t_1$ and $x(t_1) = 0$. Since by (**A**) $g(t, \psi) \geq 0$ for any $(t, \psi) \in [0, \infty) \times C$, from (2.1) we have that

$$\dot{x}(t) \geq -r(t)h(x(t)), \qquad 0 \leq t \leq t_1. \tag{4.3}$$

But from the comparison theorem of the differential inequalities (see [4]), we have

$$x(t) \geq y(t), \quad 0 \leq t \leq t_1,$$

where $y(t) = y(0, \varphi(0), 0)(t)$ is the positive solution of (4.1), with $c = 0$ and with the initial condition

$$y(0) = x(0) = \varphi(0) > 0.$$

Then at $t = t_1$ we get $x(t_1) \geq y(t_1) > 0$, which is a contradiction with our assumption that $x(t_1) = 0$. Hence $x(t) > 0$ for $t \in [0, \infty)$. □

Proof of Lemma 2.4 First, we prove part (i). Let $\varphi \in C_+$ be an arbitrary fixed initial function and $x(t) = x(\varphi)(t)$ be any solution of the IVP (2.1) and (2.2). Then, by Lemma 2.3, we have $x(t) > 0$ for $t \geq 0$. Let $T_1 \geq \tau$ and $u_1 > 0$ be defined by ($\mathbf{C_1}$). In virtue of condition (**B**), there exists a positive constant c such that

$$0 < h^{-1}(c) \leq u_1 \qquad \text{and} \qquad \min_{0 \leq t \leq T_1} x(t) > h^{-1}(c) > 0.$$

We show that $x(t) > h^{-1}(c)$ for all $t \geq 0$. Suppose there exists $\bar{t} > T_1$ such that $x(t) > h^{-1}(c)$ for $t \in [0, \bar{t})$ and $x(\bar{t}) = h^{-1}(c)$. Then, using ($\mathbf{C_1}$) with $u = h^{-1}(c)$, we have

$$g(\bar{t}, x_{\bar{t}}) \geq q_1(T_1, h^{-1}(c)) > c,$$

therefore

$$\dot{x}(\bar{t}) = r(\bar{t})\Big(g(\bar{t}, x_{\bar{t}}) - h(x(\bar{t}))\Big) > r(\bar{t})\Big(c - h(h^{-1}(c))\Big) = 0.$$

This is a contradiction, since $\dot{x}(\bar{t}) \leq 0$. Hence $x(t) > h^{-1}(c)$ holds for all $t \geq 0$, so part (i) is proved.

The proof of part (ii) is similar. □

Proof of Theorem 2.5 First, we prove part (i). Let $x(t)$ be any solution of the IVP (2.1) and (2.2), and let $T \geq \tau$. By virtue of (2.3) we have for any $T \geq \tau$

$$0 < a_{T-\tau} := \inf_{t \geq T-\tau} x(t). \tag{4.4}$$

Thus, from (4.4) and $(\mathbf{D_1})$, we get

$$g(t, x_t) \geq q_1(T, a_{T-\tau}), \qquad t \geq T.$$

Hence, from (2.1), it follows

$$\dot{x}(t) \geq r(t)[q_1(T, a_{T-\tau}) - h(x(t))], \qquad t \geq T. \tag{4.5}$$

From (4.5) and the comparison theorem of differential inequalities we see that

$$x(t) \geq y(t) \qquad \text{for } t \geq T,$$

where $y(t) = y(T, x(T), q_1(T, a_{T-\tau}))(t)$ is the solution of Eq. (4.1) with $c = q_1(T, a_{T-\tau})$ and with the initial condition

$$y(T) = x(T).$$

From Lemma 4.1, we see that

$$y(\infty) := \lim_{t\to\infty} y(t) = h^{-1}(q_1(T, a_{T-\tau})).$$

Thus

$$h^{-1}(q_1(T, a_{T-\tau})) = y(\infty) \leq \underline{x}(\infty),$$

and from the last inequality, we have

$$\liminf_{T\to\infty} h^{-1}(q_1(T, a_{T-\tau})) \leq \underline{x}(\infty).$$

But since

$$\underline{x}(\infty) = \lim_{T\to\infty} a_T,$$

then

$$\lim_{T\to\infty} a_{T-\tau} = \underline{x}(\infty). \tag{4.6}$$

Using $(\mathbf{D_1})$, (4.6) and the strict monotonicity of h^{-1}, we obtain

$$\liminf_{T\to\infty} h^{-1}(q_1(T, a_{T-\tau})) = h^{-1}(\liminf_{T\to\infty} q_1(T, a_{T-\tau})) \geq h^{-1}(q_1^*(\underline{x}(\infty))) \geq 0,$$

and hence

$$h^{-1}(q_1^*(\underline{x}(\infty))) \leq \underline{x}(\infty).$$

Therefore, the proof of (i) is completed.

The proof of part (ii) is similar to the proof of part (i), so it is omitted. □

Proof of Corollary 2.6 Theorem 2.5 yields

$$h^{-1}(q^*(\underline{x}(\infty))) \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq h^{-1}(q^*(\overline{x}(\infty))),$$

or equivalently,

$$q^*(\underline{x}(\infty)) \le h(\underline{x}(\infty)) \le h(\overline{x}(\infty)) \le q^*(\overline{x}(\infty)).$$

Then condition (2.7) implies

$$\overline{x}(\infty) \le u^* \le \underline{x}(\infty),$$

which gives (2.8). □

5. Conclusions

In this manuscript we obtained sufficient conditions for the uniform permanence of the positive solutions of a large class of nonlinear differential equations with delays of the form

$$\dot{x}(t) = r(t)\Big(g(t, x_t) - h(x(t))\Big). \tag{5.1}$$

Eq. (5.1) can be considered as a population model with delayed birth term and instantaneous death term. Our results improve earlier conditions proved in the literature, since, instead the boundedness of r, we assumed

$$\int_0^\infty r(t)\,dt = \infty.$$

Consider the differential equation

$$\dot{x}(t) = r(t)[x(t-1) - x^2(t)], \qquad t \ge 0 \tag{5.2}$$

with $r(t) := \frac{1}{2t^2+11t+14}$. It is easy to check that the function

$$x(t) = 2 + \frac{1}{2+t}, \qquad t \ge 0$$

is a positive solution of Eq. (5.2), and $\lim_{t\to\infty} x(t) = 2$. On the other hand,

$$\int_0^\infty r(t) < \infty \tag{5.3}$$

holds in this example, so our estimates do not work for Eq. (5.2). It is an interesting future work to obtain lower and upper estimates of the solutions of Eq. (5.1) in the case when (5.3) holds.

We have also given explicit conditions implying that for all $0 < p < q$, $q \ge 1$ all positive solutions of the equation

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t) x^p(t - \tau_k(t)) - \beta(t) x^q(t)$$

are asymptoticall equivalent in the sense that difference of any two positive solutions tend to 0. In particular, if the equation has a positive periodic solution, then it is unique, and it attracts all positive solutions. It is an open question to extend this result to larger classes of nonlinear delay equations.

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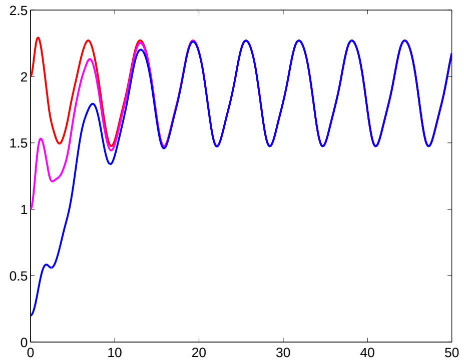

Figure 1: Solutions of Eq. (3.16) corresponding to the initial functions $\varphi(t) = 0.2$, $\varphi(t) = 1$ and $\varphi(t) = 2$

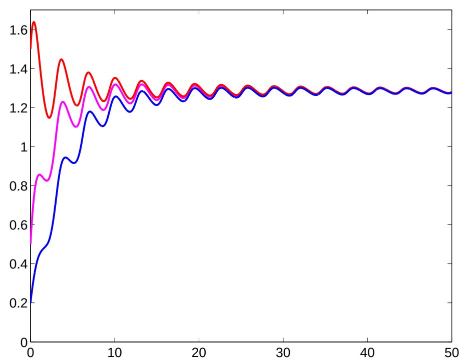

Figure 2: Solutions of Eq. (3.45) corresponding to $\delta = 0.8$ and the initial functions $\varphi(t) = 0.2$, $\varphi(t) = 0.5$ and $\varphi(t) = 1.5$