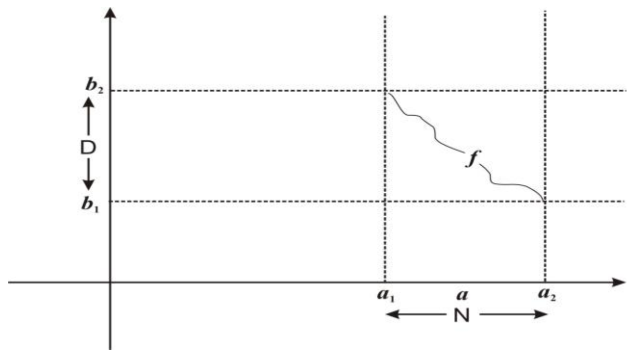
Chapter 2

Limits and continuity

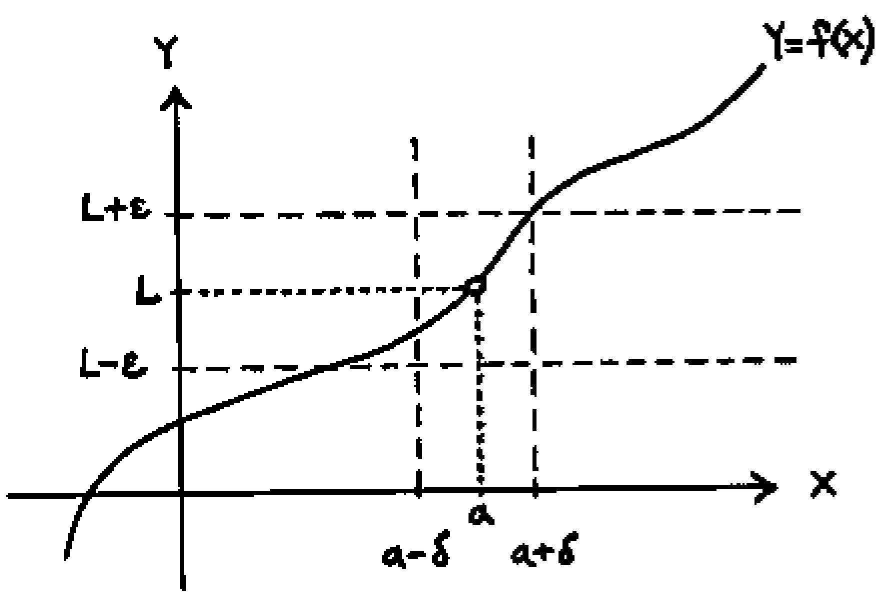
Limits and continuity

Formal Definition of a Limit:

Geometrically, the definition means that for any lines $y = b_1$, $y = b_2$ below and above the line $y = L$, there exist vertical lines $x = a_1$, $x = a_2$ to the left and right of $x = a$ so that the graph of $f(x)$ between $x = a_1$ and $x = a_2$ lies between the lines $y = b_1$ and $y = b_2$. The key phrase in the above statement is "for every open interval D", which means that even if D is very, very small (that is, $f(x)$ is very, very close to L), it still is possible to find interval N where $f(x)$ is defined for all values except possibly $x = a$.



We say that the limit of a function $f(x)$ at a is L written as $\lim_{x \to a} f(x) = L$, if for every
$\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon \Rightarrow |x - a| < \delta$.



Example 1: Use the definition of a limit to prove that

$$\lim_{x\to 3}(2x+1) = 7.$$

Solution

Let $\varepsilon > 0$ such that $|2x+1-7| < \varepsilon \Rightarrow |2x-6| < \varepsilon$

$$\Rightarrow |2(x-3)| < \varepsilon$$

$$\Rightarrow |x-3| < \frac{\varepsilon}{2}$$

Hence if we take $\delta = \frac{\varepsilon}{2}$, we have the result.

Exercises: Use the definition of a limit to prove that

1. $\lim_{x\to 1}(3x-1) = 2.$
2. $\lim_{x\to 0}(6x+5) = 5.$
3. $\lim_{x\to -1}(3x+4) = 1.$

Evaluating Limits

In this section we will continue our discussion of limits and focus on ways to evaluate limits. We will observe the limits of a few basic functions and then introduce a set of laws for working with limits. We will conclude the lesson with a theorem that will allow us to use an indirect method to find the limit of a function.

Direct Substitution and Basic Limits

Let's begin with some observations about limits of basic functions. Consider the following limit problems:

$$\lim_{x\to 2} 5\,, \quad \lim_{x\to 4} x.$$

These are examples of limits of basic constant and linear functions, $f(x) = c$ and $f(x) = mx + b$.

We note that each of these functions are defined for all real numbers. If we apply our techniques for finding the limits we see that

$$\lim_{x\to 2} 5 = 5\,,\ \lim_{x\to 4} x = 4,$$

and observe that for each the limit equals the value of the function at the x–value of interest:

$$\lim_{x\to 2} 5 = f(5) = 5\,,\ \lim_{x\to 4} x = f(4) = 4.$$

Hence $\lim_{x\to a} f(x) = f(a)$. This will also be true for some of our other basic functions, in particular all polynomial and radical functions, provided that the function is defined at $x = a$. For example, $\lim_{x\to 3} x^3 = f(3) = 27$ and $\lim_{x\to 4} \sqrt{x} = f(4) = 2$. The properties of functions that make these facts true will be discussed later. For now, we wish to use this idea for evaluating limits of basic functions. However, in order to evaluate limits of more complex function we will need some properties of limits, just as we needed laws for dealing with complex problems involving exponents. A simple example illustrates the need we have for such laws.

Example 1:

Evaluate $\lim_{x\to 2}\left(x^3 + \sqrt{2x} \right)$. The problem here is that while we know that the limit of each individual function of the sum exists, $\lim_{x\to 2} x^3 = 8$ and $\lim_{x\to 2} \sqrt{2x} = 2$, our basic limits above do not tell us what happens when we find the limit of a sum of functions. We will state a set of properties for dealing with such sophisticated functions.

<u>Some rules</u>

Let $\displaystyle \mathop{Lim}_{x\to a} f(x) = h$ $\qquad \displaystyle \mathop{Lim}_{x\to a} g(x) = k$ and a constant c, then

1) $\displaystyle \mathop{Lim}_{x\to a} c\ f(x) = c \mathop{Lim}_{x\to a} f(x)$

2) $\displaystyle \mathop{Lim}_{x\to a} \left[f(x) \pm g(x) \right] = \mathop{Lim}_{x\to a} f(x) \pm \mathop{Lim}_{x\to a} g(x)$

3) $\displaystyle \mathop{Lim}_{x\to a} \left[f(x) \cdot g(x) \right] = \mathop{Lim}_{x\to a} f(x) \cdot \mathop{Lim}_{x\to a} g(x)$

4) $\displaystyle \mathop{Lim}_{x\to a} \frac{f(x)}{g(x)} = \frac{\mathop{Lim}_{x\to a} f(x)}{\mathop{Lim}_{x\to a} g(x)}$, $\qquad \displaystyle \mathop{Lim}_{x\to a} g(x) \neq 0$

5) $\displaystyle \mathop{Lim}_{x\to a} (f(x))^n = (\mathop{Lim}_{x\to a} f(x))^n$, n is real

6) $\underset{x \to a}{Lim} \dfrac{x^m - a^m}{x - a} = m\, a^{m-1}$, m is rational

6) $\underset{x \to a}{Lim} \dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n} a^{m-n}$, m &n are rational numbers.

7) $\underset{x \to 0}{Lim} \dfrac{\sin x}{x} = 1$

8) $\underset{x \to 0}{Lim} \dfrac{\tan x}{x} = 1$

With these properties we can evaluate a wide range of polynomial and radical functions. Recalling our example above, we see that

$$\lim_{x \to 2}\left(x^3 + \sqrt{2x}\, \right) = \lim_{x \to 2}(x^3) + \lim_{x \to 2}\left(\sqrt{2x}\, \right) = 8 + 2 = 10.$$

Example 2:

Find the following limit if it exists

$$\lim_{x \to -4}\left(2x^2 - \sqrt{-x}\, \right).$$

Since the limit of each function within the parentheses exists, we can apply our properties and find

$$\lim_{x \to -4}\left(2x^2 - \sqrt{-x}\, \right) = \lim_{x \to -4} 2x^2 - \lim_{x \to -4} \sqrt{-x}.$$

Observe that the second limit, $\lim_{x \to -4} \sqrt{-x}$, is an application of Law #2 with $n = \frac{1}{2}$. So we have $\lim_{x \to -4}\left(2x^2 - \sqrt{-x}\, \right) = \lim_{x \to -4} 2x^2 - \lim_{x \to -4} \sqrt{-x} = 32 - 2 = 30.$

In most cases of sophisticated functions, we simplify the task by applying the Properties as indicated. We want to examine a few exceptions to these rules that will require additional analysis.

Strategies for Evaluating Limits of Rational Functions

Let’s recall our example

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

We saw that the function did not have to be defined at a particular value for the limit to exist. In this example, the function was not defined for $x = 1$. However we were able to evaluate the limit numerically by checking functional values around $x = 1$and found

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Note that if we tried to evaluate by direct substitution, we would get the quantity ${}^{0}\!/_{0}$, which we refer to as an ***indeterminate form*.** In particular, Property 4 for finding limits does not apply since $\lim_{x\to 1}(x - 1\ ) = 0$. Hence in order to evaluate the limit without using numerical or graphical techniques we make the following observation. The numerator of the function can be factored, with one factor common to the denominator, and the fraction simplified as follows:

$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1.$$

In making this simplification, we are indicating that the original function can be viewed as a linear function for x values close to but not equal to 1, that is, $\frac{x^2-1}{x-1} = x + 1$. for $x \neq 1$. In terms of our limits, we can say

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1}(x + 1) = 1 + 1 = 2.$$

Example 2: **F**ind $\lim_{x\to 0} \frac{x^2+5}{x}$.

This is another case where direct substitution to evaluate the limit gives the indeterminate form ${}^{0}\!/_{0}$. Reducing the fraction as before gives:

$$\lim_{x\to 0} \frac{x^2+5}{x} = \lim_{x\to 0}(x + 5) = 5.$$

Example 3: Find $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.

In order to evaluate the limit, we need to recall that the difference of squares of real numbers can be factored as $x^2 - y^2 = (x + y)(x - y)$.

We then rewrite and simplify the original function as follows:

$$\frac{\sqrt{x}-3}{x-9}=\frac{\sqrt{x}-3}{(\sqrt{x}+3)(\sqrt{x}-3)}=\frac{1}{\sqrt{x}+3}.$$

Hence $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}=\lim_{x\to 9}\frac{1}{\sqrt{x}+3}=\frac{1}{6}$.

You will solve similar examples in the homework where some clever applications of factoring to reduce fractions will enable you to evaluate the limit.

Limits of Composite Functions

While we can use the Properties to find limits of composite functions, composite functions will present some difficulties that we will fully discuss in the next Lesson. We can illustrate with the following examples, one where the limit exists and the other where the limit does not exist.

Example 4:
Consider $f(x)=\frac{1}{x+1}$, $g(x)=x^2$. Find $\lim_{x\to -1}(fog)(x)$.
We see that $(fog)(x)=\frac{1}{x^2+1}$ and by direct substitution we have
$\lim_{x\to -1}(fog)(x)=\frac{1}{(-1)^2+1}=\frac{1}{2}$.
Example 5:
Consider $f(x)=\frac{1}{x+1}$, $g(x)=-1$. Then we have that $f\big(g(x)\big)$ is undefined and we get the indeterminate form ${}^{1}/_{0}$. Hence $\lim_{x\to -1}(fog)(x)$ does not exist.

Squeeze (or, sandwich) Theorem

Suppose that $f(x)\le g(x)\le h(x)$ for x near a, and $\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L$. Then $\lim_{x\to a}g(x)=L$.

In other words, if we can find bounds for a function that have the same limit, then the limit of the function that they bound must have the same limit.

Limits of Trigonometric Functions

<u>Trigonometric Functions:</u> If c is any real number in the domain of the given function, then

$$\lim_{x\to c} \cos x = \cos c\,, \quad \lim_{x\to c} \sin x = \sin c\,, \quad \lim_{x\to c} \tan x = \tan c,$$
$$\lim_{x\to c} \sec x = \sec c\,, \quad \lim_{x\to c} \csc x = \csc c\,, \quad \lim_{x\to c} \cot x = \cot c.$$

<u>Example:</u>
Compute the limit of $f(x) = \frac{x}{\sin x - 2\cos x}$ at $x = \pi$.
We have

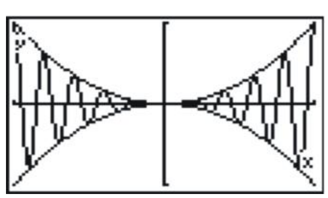
$$\lim_{x\to\pi} \frac{x}{\sin x - 2\cos x} = \frac{\pi}{\sin\pi - 2\cos\pi} = \frac{\pi}{2}$$

<u>Special Trigonometric Limits:</u>

This topic states the limits of the trigonometric functions and also two very useful limits involving sine and cosine. These special limits can either be proved with L'Hopital's rule or with the squeeze rule. Examples are given which illustrate there usefulness. The following trigonometric limits hold:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad and \quad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

Example 7: Find $\lim_{x\to 0} x^2 \cos(10\pi x)$.



From the graph we note that:

1. The function is bounded by the graphs of $x^2$and $-x^2$
2. $\lim_{x\to 0} x^2 = \lim_{x\to 0}(-x^2) = 0$.

Hence the Squeeze Theorem applies and we conclude that $\lim_{x\to 0} x^2\cos(10\pi x) = 0$.

Review Questions

Find each of the following limits if they exist.

1. $\lim_{x\to 2}(x^2 - 3x + 4)$ & $\lim_{x\to 4}\frac{x^2-16}{x-4}$ & $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$ & $\lim_{x\to -1}\frac{x-2}{x+1}$
2. $\lim_{x\to -1}\frac{10x-2}{3x+1}$ & $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$ & $\lim_{x\to 5}\frac{x^2-25}{x^3-125}$
3. Consider $f(x) = \frac{1}{x+1}$, $g(x) = x^2$. We found $\lim_{x\to -1}(fog)(x) = \frac{1}{2}$. Find $\lim_{x\to -1}(gof)(x)$.
4. Consider function $f(x)$ such that $5x - 11 \le f(x) \le x^2 - 4x + 9$ for $x \ge 0$. Use the Squeeze Theorem to find $\lim_{x\to 5} f(x)$.
5. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right) = 0$.

Review Answers

1. $\lim_{x\to 2}(x^2 - 3x + 4) = 2$ & $\lim_{x\to 4}\frac{x^2-16}{x-4} = 8$ & $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \frac{1}{4}$ & $\lim_{x\to -1}\frac{x-2}{x+1}$ does not exist.
2. $\lim_{x\to -1}\frac{10x-2}{3x+1} = 6$ & $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1} = \frac{1}{4}$ & $\lim_{x\to 5}\frac{x^2-25}{x^3-125} = \frac{2}{15}$
3. $\lim_{x\to -1}(gof)(x)$. does not exist since $g(f(x))$ is undefined.
4. $\lim_{x\to 5} f(x) = 14$ since $\lim_{x\to 5}(5x - 11) = \lim_{x\to 5}(x^2 - 4x + 9) = 14$.
5. Note that $x^4 \ge \sin\left(\frac{1}{x}\right) \ge -x^4$, and since $\lim_{x\to 0} x^4 = \lim_{x\to 0}(-x^4) = 0$, , then by the Squeeze Theorem we must have $\lim_{x\to 0} x^4\sin\left(\frac{1}{x}\right) = 0$.

Examples

Find the following limits:

2) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$ 1) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\tan 5x}$

4) $\lim\limits_{x\to 0} \dfrac{1-\cos 2x + \tan^2 x}{x \sin x}$ 3) $\lim\limits_{x\to 0} \dfrac{\cos 5x - \cos x}{x^2}$

Solutions

1) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\tan 5x} = \lim\limits_{x\to 0} \dfrac{\sin 3x}{x} \cdot \dfrac{x}{\tan 5x}$

$$= \frac{3 \lim\limits_{x\to 0} \dfrac{\sin 3x}{3x}}{5 \lim\limits_{x\to 0} \dfrac{\tan 5x}{5x}} = \frac{3}{5}$$

2) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{\sin x \left(\dfrac{1}{\cos x} - 1 \right)}{x^3}$

$$= \lim_{x\to 0} \frac{\sin x \left(\dfrac{1-\cos x}{\cos x} \right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1-\cos x}{x^2} \cdot \lim_{x\to 0} \frac{1}{\cos x}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\sin^2 (x/2)}{x^2}$$

$$= \lim_{x\to 0} \frac{2\sin^2 (x/2)}{(x/2)^2 \cdot 4} = 2 \cdot \frac{1}{4} = \frac{1}{2}$$

3) $\lim\limits_{x\to 0} \dfrac{\cos 5x - \cos x}{x^2}$

By using the property

$$\cos ax - \cos bx = -2 \sin \frac{(a+b)x}{2} \sin \frac{(a-b)x}{2}$$

3) $\lim\limits_{x\to 0} \dfrac{\cos 5x - \cos x}{x^2} = \lim\limits_{x\to 0} \dfrac{-2 \sin 3x \sin 2x}{x^2}$

$$= -2 \lim_{x\to 0} \frac{3 \sin 3x}{3x} \cdot \lim_{x\to 0} \frac{2 \sin 2x}{2x} = -12$$

$$4)\ \lim_{x\to 0} \frac{1-\cos 2x+\tan^2 x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos 2x}{x\sin x}$$

$$+\lim_{x\to 0} \frac{\tan^2 x}{x\sin x} = \lim_{x\to 0} \frac{2\sin^2 x}{x\sin x} + \lim_{x\to 0} \frac{\sin x}{x\cos^2 x} 2+1=3$$

2 Find the limit when x tends to 3

$$f(x)=\begin{cases} x^3-1 & x\le 3 \\ 8x+2 & x>3 \end{cases}$$

Solution

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^3-1) = 27-1=26$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (8x+2) = 24+2=26$$

So we have $\lim_{x\to 3} f(x)=26$

Exercises

Find the limits:

$$\lim_{x\to 0}\frac{\sin^2 x}{2x} \quad \& \quad \lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{2x^2} \quad \& \quad \lim_{x\to 0}\frac{\sin x^2\,(1+\cos x)}{2x^2} \quad \& \quad \lim_{x\to 0}\frac{\sin 5x}{\sin 4x}$$

Evaluate:

1. $\lim_{x\to \pi/2}\frac{\sin x}{x}$
2. $\lim_{x\to 0}\frac{\sin x}{2x}$
3. $\lim_{x\to 0}\frac{4(1-\cos x)}{x}$
4. $\lim_{x\to 0}\frac{\cos x\tan x}{x}$
5. $\lim_{x\to 0}\frac{\sin 3x}{3x}$
6. $\lim_{x\to 0}\frac{\sin 3x}{2x}$
7. $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}$
8. $\lim_{x\to 0}\frac{\sin 5x}{x}$
9. $\lim_{x\to 0}(1+\cos 4x)$
10. $\lim_{x\to 0}\frac{\tan x}{x}$
11. $\lim_{x\to 0}\frac{\cos 2x}{2}$

12. $f(x)=\begin{cases} \sin x & x<\dfrac{\pi}{6} \\ \dfrac{\sin x}{x} & \dfrac{\pi}{6}\le x<\pi \\ 1 & x=\pi \\ \cos x & x>\pi \end{cases}$

a) $\lim\limits_{x\to\frac{\pi}{6}} f(x)$ b) $\lim\limits_{x\to\pi} f(x)$ c) $\lim\limits_{x\to\frac{\pi}{4}} f(x)$ d) $\lim\limits_{x\to 2\pi} f(x)$

e) $f(\pi)$ f) $f\left(\dfrac{\pi}{6}\right)$

Find the following limits

$b)\ \underset{x\to 0}{Lim}\ \dfrac{5x-4\tan x}{x}$ $a)\ \underset{x\to 2}{Lim}\ \dfrac{2x^2+3x-14}{x^2-3x+2}$

$d)\ \underset{x\to 1}{Lim}\ \dfrac{x^8-1}{\sqrt[5]{x}+1}$ $c)\ \underset{x\to 2}{Lim}\ \dfrac{x^4-16}{\sqrt{x+2}-2}$

$f)\ \underset{x\to 0}{Lim}\ \dfrac{\sin 7x}{\sin 3x}$ $e)\ \underset{x\to 0}{Lim}\ \dfrac{\tan 5x}{x\cos 3x}$

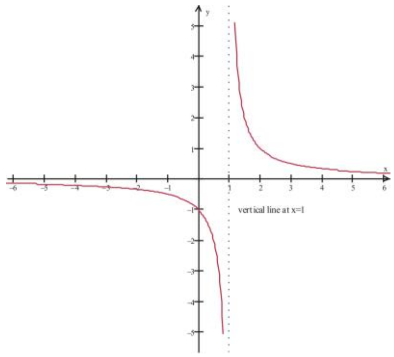
$h)\ \underset{x\to 0}{Lim}\ \dfrac{1-\cos 2x}{1-\cos 4x}$ $g)\ \underset{x\to 3}{Lim}\ \dfrac{\sqrt{x+6}-3}{\sqrt{x+1}-2}$

$j)\ \underset{x\to 0}{Lim}\ \dfrac{x^4-16}{x^3-8}$ $i)\ \underset{x\to \pi/2}{Lim}\ \dfrac{x-(\pi/2)}{\cos x}$

Infinite Limits

In this lesson we will discuss infinite limits. In our discussion the notion of infinity is discussed in two contexts. First, we can discuss infinite limits in terms of the value a function as we increase x without bound. In this case we speak of the ***limit of*** $f(x)$***as*** x ***approaches*** $\infty$and write $\lim\limits_{x\to\infty} f(x)$. We could similarly refer to the ***limit of*** $f(x)$***as*** x ***approaches*** $-\infty$ and write $\lim\limits_{x\to-\infty} f(x)$.

The second context in which we speak of infinite limits involves situations where the function values increase without bound. For example, in the case of a rational function such as $f(x)=\dfrac{x+1}{x^2-1}$, a function we discussed in previous lessons:



At $x = 1$, we have the situation where the graph grows without bound in both a positive and a negative direction. We say that we have a vertical asymptote at $x = 1$, and this is indicated by the dotted line in the graph above. In this example we note that $\lim_{x \to 1} f(x)$ does not exist. But we could compute both one-sided limits as follows.

$$\lim_{x \to 1^-} f(x) = -\infty \text{ and } \lim_{x \to 1^+} f(x) = +\infty.$$

More formally, we define these as follows:

Definition:

We observe that as x increases in the positive direction, the function values tend to get smaller. The same is true if we decrease x in the negative direction. Some of these extreme values are indicated in the following table.

x	$f(x)$
100	.0101
200	.0053
-100	$-.0099$
-200	$-.005$

Suppose we look at the function $f(x) = {(x+1)}\big/{(x^2-1)}$ and determine the infinite limits $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

The definition for negative infinite limits is similar. The right-hand limit of the function $f(x)$ at $x = a$ is infinite, and we write $\lim_{x \to a^+} f(x) = \infty$, if for every positive number k, there exists an open interval $(a\,, a + \delta)$ contained in the domain of $f(x)$, such that $f(x)$ is in $(k\,, \infty)$ for every x in $(a\,, a + \delta)$.

The following example shows how we can use this fact in evaluating limits of rational functions. Since our original function was roughly of the form $f(x) = \frac{1}{x}$, this enables us to determine limits for all other functions of the form $f(x) = \frac{1}{x^p}$ with $p > 0$. Specifically, we are able to conclude that $\lim_{x\to\infty} \frac{1}{x^p} = 0$. This shows how we can find infinite limits of functions by examining the ***end behavior*** of the function $(x) = \frac{1}{x^p}$, $p > 0$

We observe that the values are getting closer to $f(x) = 0$ Hence $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

Example 1: Find $\lim_{x\to\infty} \frac{2x^3-x^2+x-1}{x^6-x^5+3x^4-2x+1}$

Solution:

Note that we have the indeterminate form, so Limit does not hold. However, if we first divide both numerator and denominator by the quantity x^6, we will then have a function of the form

$$\frac{f(x)}{g(x)} = \frac{\frac{2x^3}{x^6} - \frac{x^2}{x^6} + \frac{x}{x^6} - \frac{1}{x^6}}{\frac{x^6}{x^6} - \frac{x^5}{x^6} + \frac{3x^4}{x^6} - \frac{2x}{x^6} + \frac{1}{x^6}} = \frac{\frac{2}{x^3} - \frac{1}{x^4} + \frac{1}{x^5} - \frac{1}{x^6}}{1 - \frac{1}{x} + \frac{3}{x^2} - \frac{2}{x^5} + \frac{1}{x^6}}$$

We observe that the limits $\lim_{x\to-\infty} f(x)$and $\lim_{x\to-\infty} g(x)$both exist. In particular, $\lim_{x\to-\infty} f(x) = 0$and $\lim_{x\to-\infty} g(x) = 1$. Hence we have $\lim_{x\to\infty} \frac{2x^3-x^2+x-1}{x^6-x^5+3x^4-2x+1} = \frac{0}{1} = 0$.

Review Questions

In problems 1 - 7, find the limits if they exist.

1. $\lim_{x\to3^+} \frac{(x+2)^2}{(x-2)^2-1}$ & $\lim_{x\to\infty} \frac{(x+2)^2}{(x-2)^2-1}$ & $\lim_{x\to1^+} \frac{(x+2)^2}{(x-2)^2-1}$ & $\lim_{x\to\infty} \frac{2x-1}{x+1}$
2. $\lim_{x\to-\infty} \frac{x^5+3x^4+1}{x^3-1}$ & $\lim_{x\to\infty} \frac{3x^4-2x^2+3x+1}{2x^4-2x^2+x-3}$ & $\lim_{x\to\infty} \frac{2x^2-x+3}{x^5-2x^3+2x-3}$

In problems 8 - 10, analyze the given function and identify all asymptotes and the end behavior of the graph.

3. $f(x) = \dfrac{(x+4)^2}{(x-4)^2-1}$

4. $f(x) = -3x^3 - x^2 + 2x + 2$

5. $f(x) = \dfrac{2x^2-8}{x+2}$

Review Answers

1. $\lim_{x\to 3^+} \frac{(x+2)^2}{(x-2)^2-1} = +\infty$ & $\lim_{x\to\infty} \frac{(x+2)^2}{(x-2)^2-1} = 1$ & $\lim_{x\to 1^+} \frac{(x+2)^2}{(x-2)^2-1} = -\infty$ & $\lim_{x\to\infty} \frac{2x-1}{x+1} = 2$
2. $\lim_{x\to-\infty} \frac{x^5+3x^4+1}{x^3-1} = -\infty$ & $\lim_{x\to\infty} \frac{3x^4-2x^2+3x+1}{2x^4-2x^2+x-3} = \frac{3}{2}$ & $\lim_{x\to\infty} \frac{2x^2-x+3}{x^5-2x^3+2x-3} = 0$
3. Zero at $x = -4$; vertical asymptotes at $x = 3, 5$; $f(x) \to 1$ as $x \to \pm\infty$
4. Zero at $x = 1$; no vertical asymptotes; $f(x) \to -\infty$ as $x \to \infty$; $f(x) \to \infty$ as $x \to -\infty$
5. Zero at $x = 2$ no vertical asymptotes but there is a discontinuity at $x = -2$ $f(x) \to -\infty$ as $x \to -\infty$.

Example (a) Show that for x > 0, $\dfrac{(x+1)(x+3)}{x+2} < \sqrt{(x+1)(x+3)} < x+2$.

(b) Hence find $\lim_{x\to\infty}[\sqrt{(x+1)(x+3)} - x]$.

N.B. $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ for any positive integer n.

Example Evaluate $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$ by sandwich rule.

Theorem $\lim_{x\to\infty}(1+\frac{1}{x})^x = e \Leftrightarrow \lim_{y\to 0}(1+y)^{\frac{1}{y}} = e$

Example Evaluate (a) $\lim_{x\to 0}(1-x)^{\frac{1}{x}}$ (b) $\lim_{x\to\infty}\left(\dfrac{x^2+1}{x^2-1}\right)^{x^2}$ (c) $\lim_{x\to 1} x^{\frac{1}{1-x}}$

N.B. 1. $\lim_{x\to-\infty}(1+\frac{1}{x})^x = e$ 2. $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$.

Exercise (a) $\lim_{x\to 0}(1-3x)^{\frac{1}{x}}$ (b) $\lim_{x\to\infty}(1+\frac{2}{x})^{-x}$

(c) $\lim_{x\to\infty}(1+\tan x)^{\cot x}$ (d) $\lim_{x\to\infty}(\frac{x+1}{x-1})^{x}$

Example Show that each of the following limits does not exist.

(a) $\lim_{x\to 2}\sqrt{x-2}$ (b) $\lim_{x\to 0}\frac{|x|}{x}$ (c) $\lim_{x\to 0}e^{\frac{1}{x}}$

Example By Sandwich rule, show that $\lim_{x\to 0}(a^{\frac{1}{x}}+b^{\frac{1}{x}})^{x}$ does not exist for $a>b>0$.

Solution If $a>b>0$ then $\frac{b}{a}<1$

If $x>0$ then $(\frac{b}{a})^{\frac{1}{x}}<1^{\frac{1}{x}}=1$

$\therefore\ a<(a^{\frac{1}{x}}+b^{\frac{1}{x}})^{x}=a\{1+(\frac{b}{a})^{\frac{1}{x}}\}^{x}<a\cdot 2^{x}$

As $\lim_{x\to 0^{+}}2^{x}=1$, by sandwich rule,

we have $\lim_{x\to 0}(a^{\frac{1}{x}}+b^{\frac{1}{x}})^{x}=a$

If $a>b>0$ and **x < 0** then

$(\frac{b}{a})^{\frac{1}{x}}>1 \Leftrightarrow (\frac{a}{b})^{\frac{1}{x}}<1$

$b<(a^{\frac{1}{x}}+b^{\frac{1}{x}})^{x}=b\{(\frac{a}{b})^{\frac{1}{x}}+1\}^{x}<b\cdot 2^{x}$

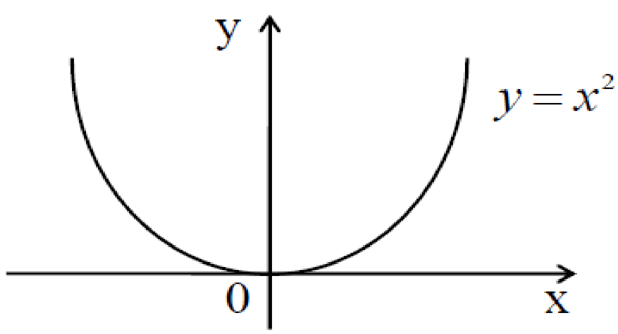
As $\lim_{x\to 0^{-}}b\cdot 2^{x}=b$, by sandwich rule,

we have $\lim_{x\to 0}(a^{\frac{1}{x}}+b^{\frac{1}{x}})^{x}=b$

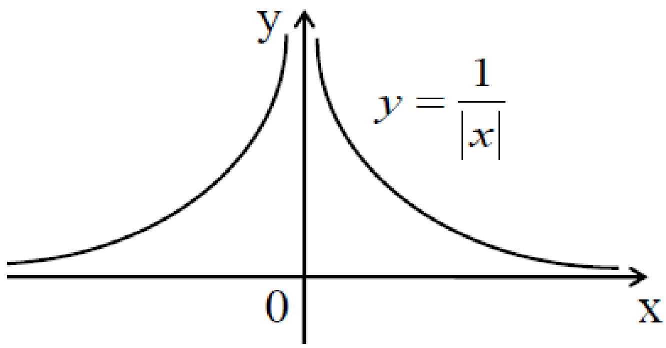
Since $a\neq b,\ \lim_{x\to 0}(a^{\frac{1}{x}}+b^{\frac{1}{x}})^{x}$ does not exist. (Why ?)

Continuity of a Function

Example $f(x)=x^{2}$ is continuous on R.



Example $f(x) = \frac{1}{|x|}$ is discontinuous at x = 0.



Continuity of a function at a point

Let *f* be a real function on a subset of the real numbers and let *c* be a point in the domain of *f.* Then *f* is continuous at *c* if

$$\lim_{x \to -c} f(x) = f(c).$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and are equal to each other, i.e.,

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

then *f* is said to be continuous at x = *c*.

Continuity in an interval

(i) *f* is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.

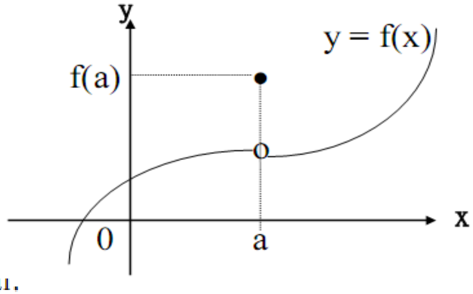
(ii) *f* is said to be continuous in the closed interval *[a, b]* if

- f is continuous in (a, b)
- $\lim_{x \to a^+} f(x) = f(a)$
- $\lim_{x \to b^-} f(x) = f(b)$

Geometrical meaning of continuity

(i) Function f will be continuous at $x = c$ if there is no break in the graph of the function at the point (c, f (c)) .

(ii) In an interval, function is said to be continuous if there is no break in the graph of the function in the entire interval.



Discontinuity

The function f will be discontinuous at $x = a$ in any of the following cases :

(i) $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist but are not equal.

(ii) $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

Example Show that $f(x) = \begin{cases} 2x+1, & x \neq 2 \\ 4 & , x = 2 \end{cases}$ is discontinuous at x = 2.

Solution Since f(2) = 4 and $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2}(2x+1) = 5 \neq f(2)$,

so f(x) is discontinuous at x = 2.

Example Let $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ a & , x = 0 \end{cases}$.

(a) Find $\lim\limits_{x \to 0} f(x)$.

(b) Find a if f(x) is continuous at x = 0.

Continuity of some of the common functions

Function $f(x)$	Interval in which f is continuous
1. The constant function, i.e. $f(x) = c$	
2. The identity function, i.e. $f(x) = x$	**R**
3. The polynomial function, i.e. $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$	
4. $\lvert x - a \rvert$	$(-\infty, \infty)$
5. x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
6. $p(x) / q(x)$, where $p(x)$ and $q(x)$ are polynomials in x	$\mathbf{R} - \{x : q(x) = 0\}$
7. $\sin x$, $\cos x$	**R**
8. $\tan x$, $\sec x$	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$
9. $\cot x$, $\operatorname{cosec} x$	$\mathbf{R} - \{(n\pi : n \in \mathbf{Z}\}$
10. e^x	**R**
11. $\log x$	$(0, \infty)$
12. The inverse trigonometric functions, i.e., $\sin^{-1} x$, $\cos^{-1} x$ etc.	In their respective domains

Continuity of composite functions

Let f and g be real valued functions such that (fog) is defined at a. If g is continuous at a and f is continuous at g (a), then (fog) is continuous at

Example 1 Find the value of the constant k so that the function f defined below is

continuous at $x = 0$, where $f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$.

Example 2 Discuss the continuity of the function $f(x) = \sin x \cos x$.

Solution Since *sin* x and *cos* x are continuous functions and product of two continuous function is a continuous function, therefore

$$f(x) = \sin x \ \cos x$$

is a continuous function.

Solution It is given that the function f is continuous at $x = 0$. Therefore, $\lim_{x\to 0} f(x) = f(0)$

$$\Rightarrow \qquad \lim_{x\to 0} \frac{1-\cos 4x}{8x^2} = k$$

$$\Rightarrow \qquad \lim_{x\to 0} \frac{2\sin^2 2x}{8x^2} = k$$

$$\Rightarrow \qquad \lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right)^2 = k$$

$$\Rightarrow \qquad k = 1$$

Thus, f is continuous at $x = 0$ if $k = 1$.

Example 3 If $f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & x \neq 2 \\ k & , x = 2 \end{cases}$ is continuous at $x = 2$, find the value of k.

Solution Given $f(2) = k$.

Now, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}$

$$= \lim_{x \quad 2} \frac{(x \quad 5)(x-2)^2}{(x-2)^2} \quad \lim_{x \quad 2}(x \quad 5) \quad 7$$

As f is continuous at $x = 2$, we have

$$\lim_{x\to 2} f(x) = f(2)$$

$$\Rightarrow \qquad k = 7.$$

Example 4 Show that the function f defined by

$$f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$.

Solution Left hand limit at $x = 0$ is given by

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x\sin\frac{1}{x} = 0 \qquad \left[\text{since, } -1 < \sin\frac{1}{x} < 1\right]$$

Similarly $\lim\limits_{x \quad 0} f(x) \quad \lim\limits_{x \quad 0} x\sin\frac{1}{x} \quad 0$. Moreover $f(0) = 0$.

Thus $\lim\limits_{x \quad 0^-} f(x) \quad \lim\limits_{x \quad 0} f(x) \quad f(0)$. Hence f is continuous at $x = 0$

Example 5 Given $f(x) = \frac{1}{x-1}$. Find the points of discontinuity of the composite function $y = f[f(x)]$.

Solution We know that $f(x) = \frac{1}{x-1}$ is discontinuous at $x = 1$

Now, for $x \quad 1$,

$$f(f(x)) \quad = f \quad \frac{1}{x-1} \quad = \frac{1}{\frac{1}{x-1} - 1} \quad \frac{x-1}{2-x},$$

which is discontinuous at $x = 2$.
Hence, the points of discontinuity are $x = 1$ and $x = 2$.

Exercises

For what values of x are each of the following functions discontinuous?

a) $f(x) = \frac{1}{x}$ b) $g(x) = \frac{2}{x-2}$ c) $h(x) = \frac{x+1}{x^2-1}$

d) $i(x) = \tan(x)$ e) $j(x) = \frac{1}{\cos x - 1}$ f) $k(x) = \frac{x+2}{x^2+5}$

g) $l(x) = \frac{x+4}{x+4}$ and $l(-4) = 1$

Find out whether the given function is continuous or discontinuous :

a) $y = \begin{cases} \dfrac{x^2-9}{x+3} & x \neq -3 \\ 6 & x = -3 \end{cases}$

b) $y = \begin{cases} \dfrac{x^3-4x}{x-2} & x \neq 2 \\ 8 & x = 2 \end{cases}$

c) $y = \begin{cases} 3+x^2 & x \leq 0 \\ \dfrac{\sin 3x}{x} & x > 0 \end{cases}$

d) $y = \begin{cases} x^2 & x \in (-\infty;1\rangle \\ 2x & x \in (1;\infty) \end{cases}$

e) $y = \begin{cases} \dfrac{x^3-1}{1-x} & x \neq 1 \\ -3 & x = 1 \end{cases}$

f) $y = \begin{cases} \arctan x & x \in (-\infty;0\rangle \\ x^3 & x \in (0;1\rangle \\ 2-x & x \in (1;\infty) \end{cases}$

g) $y = \dfrac{1}{e^x - 1}$

h) $y = \begin{cases} \dfrac{2x-8}{\sqrt{x-1}-\sqrt{3}} & x \neq 4 \\ 2 & x = 4 \end{cases}$

i) $y = \begin{cases} x+1 & x \in (-\infty;0\rangle \\ 2-x^2 & x \in (0;\infty) \end{cases}$

j) $y = \begin{cases} 3-x & x \in (-\infty;2\rangle \\ (x-1)^2 & x \in (2;\infty) \end{cases}$

k) $y = \begin{cases} x^3+1 & x \in (-\infty;0\rangle \\ \sqrt{x} & x \in (0;\infty) \end{cases}$

l) $y = \begin{cases} \dfrac{2x^2+6x-20}{x^3-3x^2+2x} & x \neq 2, x \neq 0, x \neq 1 \\ 7 & x = 2 \end{cases}$

m) $y = \begin{cases} x \arctan \dfrac{1}{x} & x < 0 \\ 0 & x = 0 \\ \dfrac{1-\cos x}{x^2} & x > 0 \end{cases}$

n) $y = \operatorname{sgn} \sin x$