The geometry of Finsler spaces of H_p-scalar curvature

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Dedicated to Professor Nabil Youssef on the occasion of his 70th birthday

Abstract. The aim of the present paper is to provide an *intrinsic* investigation of special Finsler spaces of H_p-scalar curvature and of H_p-constant curvature. Characterizations of such spaces are given. Sufficient condition for Finsler space of H_p-scalar curvature to be of perpendicular scalar curvature is investigated. Necessary and sufficient condition under which a Finsler space of scalar curvature turns into a Finsler space of H_p-scalar curvature is shown. Further, certain conditions under which a Finsler manifolds of H_p-scalar curvature and of scalar curvature reduce to a Finsler space of H_p-constant curvature are obtained. Finally, various examples are studied and constructed.

1 Introduction

In Riemannian geometry, the study of the Riemannian manifolds of scalar curvature was very fruitful. It has been contributed in classifying lots of Riemannian manifolds. For example, the special manifolds of constant curvature $-1, 0, +1$. The concept of scalar curvature was extended to Finsler geometry. Most of the special spaces are derived from the fact that the tensor fields (torsions and curvatures) associated with a linear connection can be given in special forms. In Riemannian geometry there exist a unique linear connection, that is, Levi-Civita connection. But in Finsler geometry there are lots of linear connections, for example, Cartan connection , Berwald connection, ...etc. Consequently, the special Finsler spaces are more numerous than those of Riemannian geometry. Special Finsler spaces are investigated locally (using local coordinates) by many authors, see for example ([2], [6], [7], [8], [9], [10], [11], [17]). On the other hand, the global (or intrinsic, free from local coordinates) investigation of such spaces is very rare in the literature. Some considerable contributions in this direction are [16], [22] and [19].

In a recent paper [22], some characterizations of a Finsler space of scalar curvature are investigated. Also, necessary and sufficient conditions under which a Finsler space of scalar curvature reduces to a Finsler space of constant curvature are shown.

The present paper is a continuation of [22] and [19], where we provide an *intrinsic* investigation of some important special Finsler manifolds related to the Berwald curvature tensors namely, Finsler manifold of H_p-scalar curvature and of H_p-constant curvature.

In [17], Yoshida introduced, locally, the notion of Finsler space of H_p-scalar curvature. In this paper, we study, in a coordinate-free fashion, the Finsler spaces of H_p-scalar curvature and H_p-constant curvature. We give a characterization for any Finsler manifold to be of H_p-scalar curvature. We find a sufficient condition under which a Finsler space of H_p-scalar curvature is of perpendicular scalar curvature.

Section 3 is devoted to focus on the Finsler spaces of scalar curvature and constant curvature. Necessary and sufficient condition under which a Finsler space of scalar curvature is of H_p-scalar curvature is given. We show that every Finsler space of constant curvature is of H_p-scalar curvature. The converse is true only in some specific cases, for example, see Theorem 3.6. But, generally, not every Finsler space of H_p-constant curvature is of Constant curvature, see Section 4 (Example 6).

In Section 4, we study various examples. Some of these examples are mentioned in the literature, but in different contexts. And some examples are constructed, for instance Example 5.

It should finally be noted that the present work is formulated in a prospective modern coordinate-free form. Moreover, the outcome of this work is twofold. Firstly, the local expressions of the obtained results, when calculated, coincide with the existing local results. Secondly, new coordinates-free proofs have been established.

2   Notations and preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1, 13, 14, 15, 16].

Throughout, M is a smooth manifold of finite dimension n. The $\mathbb{R}$-algebra of smooth real-valued functions on M is denoted by $C^\infty(M)$; $\mathfrak{X}(M)$ stands for the $C^\infty(M)$-module of vector fields on M. The tangent bundle of M is $\pi_M : TM \longrightarrow M$, the subbundle of nonzero tangent vectors to M is $\pi : \mathcal{T}M \longrightarrow M$. The vertical subbundle of TTM is denoted by $V(TM)$. The pull-back of TM over π is $P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$.

The sections of $\pi^{-1}(TM)$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundle morphisms:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0.$$

Here $\rho := (\pi_{\mathcal{T}M}, \pi_*)$, and γ is defined by $\gamma(u, v) := j_u(v)$, where j_u is the canonical isomorphism from $T_{\pi_M(v)}M$ onto $T_u(T_{\pi_M(v)}M)$. Then, $J := \gamma \circ \rho$ is a vector 1-form

on TM called the vertical endomorphism. The Liouville vector field on TM is the vector field defined by $C := \gamma \circ \overline{\eta}$, $\overline{\eta}(u) = (u, u)$, $u \in TM$.

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. The connection (or the deflection) map associated with D is defined by

$$K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}.$$

A tangent vector $X \in T_u(\mathcal{T}M)$ at $u \in \mathcal{T}M$ is horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ is called the horizontal space at u. The connection D is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \qquad \forall u \in \mathcal{T}M.$$

Let $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$, called the horizontal map of the connection D, then

$$\rho \circ \beta = id_{\pi^{-1}(TM)}, \qquad \beta \circ \rho = id_{H(\mathcal{T}M)} \text{ on } H(\mathcal{T}M).$$

For a regular connection D, the horizontal covariant derivative $\overset{h}{D}$ and the vertical covariant derivatives $\overset{v}{D}$ are defined, for a vector $(1)\pi$-form A, for example, by

$$(\overset{h}{D} A)(\overline{X}, \overline{Y}) := (D_{\beta\overline{X}}A)(\overline{Y}), \quad (\overset{v}{D} A)(\overline{X}, \overline{Y}) := (D_{\gamma\overline{X}}A)(\overline{Y}).$$

The (classical) torsion tensor $\mathbf{T}$ (resp. the curvature tensor $\mathbf{K}$) of the connection D are given by

$$\mathbf{T}(X, Y) = D_X\rho Y - D_Y\rho X - \rho[X, Y],$$
$$\mathbf{K}(X, Y)\rho Z = -D_X D_Y \rho Z + D_Y D_X \rho Z + D_{[X,Y]}\rho Z,$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$. The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors are defined respectively by

$$Q(\overline{X}, \overline{Y}) := \mathbf{T}(\beta\overline{X}, \beta\overline{Y}), \quad T(\overline{X}, \overline{Y}) := \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}) \quad \forall \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

and the horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors are defined respectively by

$$R(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X}, \beta\overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z}, \quad S(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z}.$$

The (v)h-, (v)hv- and (v)v-torsion tensors are defined respectively by

$$\widehat{R}(\overline{X}, \overline{Y}) := R(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) := P(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) := S(\overline{X}, \overline{Y})\overline{\eta}.$$

For a Finsler manifold (M, L), we shall denote by D° the Berwald connection on the pullback bundle.

Theorem 2.1. [18] *Let (M, L) be a Finsler manifold. There exists a unique regular connection D° on $\pi^{-1}(TM)$ such that*

(a) $D^\circ_{h\circ X} L = 0$,

(b) D° *is torsion-free*: $\mathbf{T}^\circ = 0$,

(c) *The (v)hv-torsion tensor $\widehat{P^\circ}$ of D° vanishes*: $\widehat{P^\circ}(\overline{X}, \overline{Y}) = 0$.

Such a connection is called the Berwald connection associated with the Finsler manifold (M, L).

3 Finsler spaces of H_p-scalar curvature

In this section, we investigate (intrinsically) some important special Finsler spaces related to the Berwald curvature tensors namely, Finsler spaces of H_p-scalar curvature and of H_p- constant curvature. Characterizations of such spaces are obtained. Relation between Finsler spaces of H_p-scalar curvature and of perpendicular scalar curvature is investigated.

Throughout, for given Finsler manifold (M, L), g denotes the Finsler metric on $\pi^{-1}(TM)$ and ∇ denotes the Cartan connection. Also, T stands for the Cartan tensor, R and $\widehat{R}$ for the h-curvature and $(v)h$-torsion of Cartan connection, $\overset{\circ}{R}$ and $\overset{\widehat{\circ}}{R}$ for the h-curvature and $(v)h$-torsion of Berwald connection, and $H := i_{\overline{\eta}}\overset{\widehat{\circ}}{R}$ for the deviation tensor. Moreover, $\ell := L^{-1}i_{\overline{\eta}}g$, $\phi(\overline{X}) := \overline{X} - L^{-1}\ell(\overline{X})\overline{\eta}$ and $\hbar(\overline{X}, \overline{Y}) := g(\phi(\overline{X}), \overline{Y}) = g(\overline{X}, \overline{Y}) - \ell(\overline{X})\ell(\overline{Y})$ the angular metric tensor. Finally, $\overset{h}{D}{}^{\circ}$ and $\overset{v}{D}{}^{\circ}$ will denote respectively the horizontal covariant derivative and the vertical covariant derivative associated with D°.

We begin with the following definitions and results of [22] which are useful for subsequent use.

Definition 3.1. *[19]* A Finsler manifold (M, L) of dimension $n \geq 3$ is called of scalar curvature k if the deviation tensor H satisfies

$$H(\overline{X}) = kL^2\phi(\overline{X}),$$

where $k(x, y) \in C^{\infty}(\mathcal{T}M)$ is a positively homogenous of degree zero in y ($x \in M$ and $y \in T_xM$). Especially, if the scalar curvature k is constant, then (M, L) is called a Finsler manifold of constant curvature.

Definition 3.2. [19] Let $\mathcal{P}$ be the projection operator of indicatrix (or simply, projection operator) and $\phi(\overline{X}) := \overline{X} - L^{-1}\ell(\overline{X})\overline{\eta}$. If ω is a π-tensor field of type *(1,p)*, then $\mathcal{P} \cdot \omega$ is a π-tensor field of the same type defined by:

$$(\mathcal{P} \cdot \omega)(\overline{X}_1, ..., \overline{X}_p) := \phi(\omega(\phi(\overline{X}_1), ..., \phi(\overline{X}_p))).$$

Similarly, if ω is a π-tensor field of type *(0,p)*, then $\mathcal{P} \cdot \omega$ is a π-tensor field of the same type defined by:

$$(\mathcal{P} \cdot \omega)(\overline{X}_1, ..., \overline{X}_p) := \omega(\phi(\overline{X}_1), ..., \phi(\overline{X}_p)).$$

Moreover, for any π-tensor field ω is called an indicatory tensor if $\mathcal{P} \cdot \omega = \omega$.

Theorem 3.1. *[22] A Finsler manifold (M, L) is of scalar curvature k if, and only, if the h-curvature $\overset{\circ}{R}$ satisfies* [1]

$$
\begin{aligned}
\overset{\circ}{R}(\overline{X}, \overline{Y})\overline{Z} &= \mathfrak{A}_{\overline{X}, \overline{Y}}\{\phi(\overline{Y})\{\ell(\overline{Z})[k\ell(\overline{X}) + \frac{1}{3}C^k(\overline{X})] + \frac{1}{3}B^k(\overline{Z}, \overline{X}) \\
&\quad + \frac{2}{3}\ell(\overline{X})C^k(\overline{Z}) + k\hbar(\overline{Z}, \overline{X})\} + \frac{1}{3}\ell(\overline{X})C^k(\overline{Y})\phi(\overline{Z}) \\
&\quad + L^{-1}\hbar(\overline{X}, \overline{Z})\overline{\eta}\,[k\ell(\overline{Y}) + \frac{1}{3}C^k(\overline{Y})]\},
\end{aligned}
$$

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[1] $\mathfrak{A}_{\overline{X}, \overline{Y}}\{\omega(\overline{X}, \overline{Y})\} := \omega(\overline{X}, \overline{Y}) - \omega(\overline{Y}, \overline{X})$

where $C^k(\overline{X}) := L(\overset{v}{D^\circ} k)(\overline{X})$, $\quad B^k(\overline{X}, \overline{Y}) := L(\mathcal{P} \cdot \overset{v}{D^\circ} C^k)(\overline{X}, \overline{Y})$, and $\overset{v}{D^\circ}$ is the vertical covariant derivative associated with D°.

Theorem 3.2. *[22] A necessary and sufficient condition for a Finsler manifold of scalar curvature k to be of constant curvature is that the π-scalar form C^k (or B^k) vanishes.*

Remark 3.3. One easily show that the π-tensor fields $\widehat{P}$, T, ϕ, $\hbar$, C^k and B^k are indicatory tensors, and that $\mathcal{P} \cdot \ell$ vanishes identically.

Now, we are in a position to introduce the definition of Finsler spaces of H_p-scalar curvature.

Definition 3.4. A Finsler manifold (M, L) of dimension $n \geq 4$ is called of H_p-scalar curvature (or, simply H_p-sc) ε if the h-curvature tensor of Berwald connection $\overset{\circ}{R}$ satisfies

$$(\mathcal{P} \cdot \overset{\circ}{R})(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \varepsilon \, \mathfrak{A}_{\overline{X}, \overline{Y}} \left\{ \hbar(\overline{X}, \overline{Z}) \hbar(\overline{Y}, \overline{W}) \right\},$$

where $\varepsilon(x, y) \in C^\infty(\mathcal{T}M)$ is a positively homogenous of degree zero in y. Especially, if the H_p-scalar curvature ε is constant, then (M, L) is called a Finsler manifold of H_p-constant curvature (or, simply H_p-cc). Moreover, if the H_p-scalar curvature ε vanishes, then (M, L) is called a Finsler manifold of vanishing H_p-scalar curvature.

Now, we investigate intrinsically a characterization for a Finsler spaces of H_p-sc.

Theorem 3.3. *A Finsler manifold is of H_p-sc k, if and only if the h-Berwald curvature $\overset{\circ}{R}$ has the form*

$$
\begin{aligned}
\overset{\circ}{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{w}) \;=\; & L^{-1}\{\ell(\overline{Z})\widehat{R}(\overline{X}, \overline{Y}, \overline{W}) - \ell(\overline{W})\widehat{R}(\overline{X}, \overline{Y}, \overline{Z})\} \\
& + \mathfrak{A}_{\overline{X}, \overline{Y}}\{L^{-1}\ell(\overline{X})\{\widehat{R}(\overline{Z}, \overline{W}, \overline{Y}) - T(H(\overline{Z}), \overline{W}, \overline{Y}) + T(H(\overline{W}), \overline{Z}, \overline{Y}) \\
\text{(3.1)} \quad & - T(H(\overline{Y}), \overline{Z}, \overline{W}) - (D^o_{\beta\overline{\eta}}\widehat{P})(\overline{Z}, \overline{W}, \overline{Y})\} \\
& - L^{-2}\{\widehat{R}(\overline{\eta}, \overline{X}, \overline{Z})\ell(\overline{Y})\ell(\overline{W}) + \widehat{R}(\overline{\eta}, \overline{Y}, \overline{W})\ell(\overline{X})\ell(\overline{Z}) - kL^2 \hbar(\overline{X}, \overline{Z})\hbar(\overline{Y}, \overline{W})\}\},
\end{aligned}
$$

where $\widehat{P}$ is the (v)hv-torsion of Cartan connection ∇, $\widehat{P}(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$ $\widehat{R}(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z})$ *and* $T(\overline{X}, \overline{Y}, \overline{Z}) := g(T(\overline{X}, \overline{Y}), \overline{Z})$.

To prove this theorem, we need the following two lemmas.

Lemma 3.4. *[20] The h-curvature tensor $\overset{\circ}{R}$ of the Berwald connection has the properties:*

(a) $\overset{\circ}{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -\overset{\circ}{R}(\overline{Y}, \overline{X}, \overline{Z}, \overline{W})$,

(b) $\widehat{\overset{\circ}{R}}(\overline{X}, \overline{Y}) = \widehat{R}(\overline{X}, \overline{Y})$,

(c) $\overset{\circ}{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) + \overset{\circ}{R}(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) = 2\mathfrak{A}_{\overline{X}, \overline{Y}}\{(D^\circ_{\beta\overline{Y}}\widehat{P})(\overline{X}, \overline{Z}, \overline{W})\} - 2T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}, \overline{W})$,

(d) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{\overset{\circ}{R}(\overline{X}, \overline{Y})\overline{Z}\} = 0$,

Lemma 3.5. *The h-curvature tensor $\overset{\circ}{R}$ of the Berwald connection satisfies*

$$\overset{\circ}{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \overset{\circ}{R}(\overline{Z},\overline{W},\overline{X},\overline{Y}) + (D^{\circ}_{\beta\overline{Y}}\widehat{P})(\overline{X},\overline{Z},\overline{W}) - (D^{\circ}_{\beta\overline{X}}\widehat{P})(\overline{Z},\overline{W},\overline{Y})$$
$$+ (D^{\circ}_{\beta\overline{Z}}\widehat{P})(\overline{X},\overline{W},\overline{Y}) - (D^{\circ}_{\beta\overline{W}}\widehat{P})(\overline{X},\overline{Z},\overline{Y}) + T(\widehat{R}(\overline{X},\overline{W}),\overline{Z},\overline{Y})$$
$$- T(\widehat{R}(\overline{Y},\overline{W}),\overline{X},\overline{Z}) + T(\widehat{R}(\overline{Y},\overline{Z}),\overline{X},\overline{W}) - T(\widehat{R}(\overline{X},\overline{Z}),\overline{Y},\overline{W})$$
$$+ T(\widehat{R}(\overline{Z},\overline{W}),\overline{Y},\overline{X}) - T(\widehat{R}(\overline{X},\overline{Y}),\overline{Z},\overline{W}).$$

Proof. The proof is a direct consequence of Lemma 3.4. $\qquad\square$

Proof of Theorem 3.3: Applying the projection operator $\mathcal{P}$ on the h-curvature tensor $\overset{\circ}{R}$ taking into account definition 3.2 and Lemma 3.4, we obtain

$$
\begin{aligned}
(\mathcal{P}\cdot\overset{\circ}{R})(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= \overset{\circ}{R}(\overline{Z},\overline{W},\overline{X},\overline{Y}) - L^{-1}\{\ell(\overline{W})\overset{\circ}{R}(\overline{X},\overline{Y},\overline{Z},\overline{\eta}) + \ell(\overline{Z})\overset{\circ}{R}(\overline{X},\overline{Y},\overline{\eta},\overline{W})\} \\
&\quad + \ell(\overline{Y})\overset{\circ}{R}(\overline{X},\overline{\eta},\overline{Z},\overline{W}) + \ell(\overline{X})\overset{\circ}{R}(\overline{\eta},\overline{Y},\overline{Z},\overline{W})\} \\
&\quad + L^{-2}\overset{\circ}{R}(\overline{X},\overline{\eta},\overline{Z},\overline{\eta})\ell(\overline{Y})\ell(\overline{W}) + \overset{\circ}{R}(\overline{X},\overline{\eta},\overline{\eta},\overline{W})\ell(\overline{Y})\ell(\overline{Z}) \\
&\quad + \overset{\circ}{R}(\overline{\eta},\overline{Y},\overline{Z},\overline{\eta})\ell(\overline{X})\ell(\overline{W}) + \overset{\circ}{R}(\overline{\eta},\overline{Y},\overline{\eta},\overline{W})\ell(\overline{X})\ell(\overline{Z})\}.
\end{aligned}
$$
(3.2)

On the other hand, from Lemma 3.4 and Lemma 3.5, we obtain

$$
\begin{aligned}
\overset{\circ}{R}(\overline{\eta},\overline{Y},\overline{Z},\overline{W}) &= \widehat{R}(\overline{Z},\overline{W},\overline{Y}) - (D^{\circ}_{\beta\overline{\eta}}\widehat{P})(\overline{Z},\overline{W},\overline{Y}) + T(H(\overline{W}),\overline{Z},\overline{Y}) \\
&\quad - T(H(\overline{Z}),\overline{Y},\overline{W}) - T(H(\overline{Y}),\overline{Z},\overline{W}). \\
\overset{\circ}{R}(\overline{\eta},\overline{Y},\overline{Z},\overline{\eta}) &= -\widehat{R}(\overline{\eta},\overline{Y},\overline{Z}) = g(H(\overline{Y}),\overline{Z}). \\
\overset{\circ}{R}(\overline{X},\overline{\eta},\overline{\eta},\overline{Y}) &= -\widehat{R}(\overline{\eta},\overline{X},\overline{Y}).
\end{aligned}
$$

By using the above relations, Equation (3.2) becomes

$$
\begin{aligned}
(\mathcal{P}\cdot\overset{\circ}{R})(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= \overset{\circ}{R}(\overline{Z},\overline{W},\overline{X},\overline{Y}) - L^{-1}\{\ell(\overline{Z})\widehat{R}(\overline{X},\overline{Y},\overline{W}) - \ell(\overline{W})\widehat{R}(\overline{X},\overline{Y},\overline{Z})\} \\
&\quad - \mathfrak{A}_{\overline{X},\overline{Y}}\{L^{-1}\ell(\overline{X})\{\widehat{R}(\overline{Z},\overline{W},\overline{Y}) - T(H(\overline{Z}),\overline{W},\overline{Y}) + T(H(\overline{W}),\overline{Z},\overline{Y}) \\
&\quad - T(H(\overline{Y}),\overline{Z},\overline{W}) - (D^{\circ}_{\beta\overline{\eta}}\widehat{P})(\overline{Z},\overline{W},\overline{Y})\} \\
&\quad - L^{-2}\{\widehat{R}(\overline{\eta},\overline{X},\overline{Z})\ell(\overline{Y})\ell(\overline{W}) + \widehat{R}(\overline{\eta},\overline{Y},\overline{W})\ell(\overline{X})\ell(\overline{Z})\}\},
\end{aligned}
$$

Now, if (M,L) is a Finsler manifold of H_p-sc k, then Equation (3.1) is satisfied.

Conversely, suppose that (M,L) is a Finsler manifold satisfying Equation (3.1). Then, by applying the projection operator $\mathcal{P}$ on Equation (3.1) taking Remark 3.3 into account, one can deduce that (M,L) is of H_p-sc k. $\qquad\square$

In view of Theorem 3.3 and Remark 3.3, we have

Corollary 3.6. *In a Finsler manifold of H_p-sc, we have*

(a) $\overset{\circ}{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) + \overset{\circ}{R}(\overline{X},\overline{Y},\overline{W},\overline{Z}) = 2L^{-1}\ell(\overline{Y})\mathfrak{A}_{\overline{X},\overline{Y}}\{(D^{\circ}_{\beta\overline{\eta}}\widehat{P})(\overline{Z},\overline{W},\overline{Y}) + T(H(\overline{X}),\overline{Z},\overline{W})\}$

(b) $(\mathcal{P}\cdot\overset{\circ}{R})(\overline{X},\overline{Y},\overline{Z},\overline{W}) + (\mathcal{P}\cdot\overset{\circ}{R})(\overline{X},\overline{Y},\overline{W},\overline{Z}) = 0$

Remark 3.5. In Definition 3.4, if we replace the h-curvature tensor $\overset{\circ}{R}$ of Berwald connection by the h-curvature tensor R of Cartan connection, then (M, L) is called of perpendicular scalar curvature ε.

Let us define the following tensor:

$$(3.3) \qquad Q(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(\widehat{P}(\overline{X}, \overline{W}), \widehat{P}(\overline{Y}, \overline{Z})) - g(\widehat{P}(\overline{X}, \overline{Z}), \widehat{P}(\overline{Y}, \overline{W})).$$

Theorem 3.7. *If* $Q(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = q\left\{\hbar(\overline{X}, \overline{Z})\hbar(\overline{Y}, \overline{W}) - \hbar(\overline{X}, \overline{W})\hbar(\overline{Y}, \overline{Z})\right\}$ *holds, then the Finsler manifold of H_p-sc k is of perpendicular scalar curvature $(k - q)$.*

Proof. By [20], we have

$$\overset{\circ}{R}(\overline{X}, \overline{Y})\overline{Z} = R(\overline{X}, \overline{Y})\overline{Z} - T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}) - \mathfrak{A}_{\overline{X}, \overline{Y}}\{(\nabla_{\beta\overline{X}}\widehat{P})(\overline{Y}, \overline{Z}) + \widehat{P}(\overline{X}, \widehat{P}(\overline{Y}, \overline{Z}))\}.$$

From which together with (3.3), taking into account the facts that $R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -(\overline{X}, \overline{Y}, \overline{W}, \overline{Z})$ and $g((\nabla_{\beta\overline{X}}\widehat{P})(\overline{Y}, \overline{Z}), \overline{W}) = g((\nabla_{\beta\overline{X}}\widehat{P})(\overline{Y}, \overline{W}), \overline{Z})$, we obtain

$$R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \frac{1}{2}\{\overset{\circ}{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) - \overset{\circ}{R}(\overline{X}, \overline{Y}, \overline{W}, \overline{Z})\} - Q(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}).$$

Applying the projection operator $\mathcal{P}$ on both sides of the above relation and using Remark 3.3, one can deduce

$$(\mathcal{P} \cdot R)(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \frac{1}{2}\{(\mathcal{P}\overset{\circ}{R})(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) - (\mathcal{P}\overset{\circ}{R})(\overline{X}, \overline{Y}, \overline{W}, \overline{Z})\} - Q(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}).$$

Now, if (M, L) is a Finsler manifold of H_p-s.c. k and using the given assumption for Q, we get

$$(\mathcal{P} \cdot R)(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = (k - q)\left\{\hbar(\overline{X}, \overline{Z})\hbar(\overline{Y}, \overline{W}) - \hbar(\overline{X}, \overline{W})\hbar(\overline{Y}, \overline{Z})\right\},$$

hence the Finsler manifold (M, L) is of perpendicular scalar curvature $(k - q)$. $\square$

4   Finsler spaces of scalar curvature and H_p-sc

Here, the necessary and sufficient condition under which a Finsler manifold of scalar curvature turns into a Finsler manifold of H_p-scalar curvature is investigated. Moreover, certain condtions under which a Finsler manifolds of H_p-scalar curvature and of scalar curvature reduces to a Finsler manifold of H_p-constant curvature or to a Finsler manifold of vanishing H_p-scalar curvature are obtained.

Theorem 4.1. *Let (M, L) be a Finsler manifold of scalar curvature k of dimension $n \geq 4$. Then, (M, L) is a Finsler manifold of H_p-scalar curvature ε if, and only if, $B^k(\overline{X}, \overline{Y}) = \alpha\hbar(\overline{X}, \overline{Y})$, where $\alpha = 3(\varepsilon - k)$.*

Proof. Suppose that (M, L) is a Finsler manifold of scalar curvature k of dimension $n \geq 4$. Hence, by Theorem 3.1, the h-curvature tensor $\overset{\circ}{R}$ has the form

$$\begin{aligned}
\overset{\circ}{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) &= \mathfrak{A}_{\overline{X}, \overline{Y}}\{\hbar(\overline{Y}, \overline{W})\{\ell(\overline{Z})[k\ell(\overline{X}) + \frac{1}{3}C^k(\overline{X})] + \frac{1}{3}B^k(\overline{Z}, \overline{X}) \\
&\quad + \frac{2}{3}\ell(\overline{X})C^k(\overline{Z}) + k\hbar(\overline{Z}, \overline{X})\} + \frac{1}{3}\ell(\overline{X})C^k(\overline{Y})\hbar(\overline{Z}, \overline{W}) \\
&\quad + L^{-1}\hbar(\overline{X}, \overline{Z})\ell(\overline{W})[k\ell(\overline{Y}) + \frac{1}{3}C^k(\overline{Y})]\}.
\end{aligned}$$

Applying the projection operator $\mathcal{P}$ on both sides of the above equation, taking Remark 3.3 into account, we have

$$(4.1) \qquad (\mathcal{P} \cdot \overset{\circ}{R})(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \mathfrak{A}_{\overline{X}, \overline{Y}} \{\hbar(\overline{Y}, \overline{W})\{\frac{1}{3} B^k(\overline{Z}, \overline{X}) + k\hbar(\overline{Z}, \overline{X})\}\}.$$

Now, if (M, L) is a Finsler manifold of H_p-scalar curvature ε, then by Definition 3.4

$$(4.2) \qquad (\mathcal{P} \cdot \overset{\circ}{R})(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \varepsilon \, \mathfrak{A}_{\overline{X}, \overline{Y}} \left\{\hbar(\overline{X}, \overline{Z})\hbar(\overline{Y}, \overline{W})\right\}.$$

Hence, from (4.1) and (4.2), we get

$$\varepsilon\hbar(\overline{Z}, \overline{X}) = \frac{1}{3} B^k(\overline{Z}, \overline{X}) + k\hbar(\overline{Z}, \overline{X})$$

Consequently,

$$(4.3) \qquad B^k(\overline{X}, \overline{Y}) = 3(\varepsilon - k)\hbar(\overline{X}, \overline{Y}) = \alpha\hbar(\overline{X}, \overline{Y}).$$

Conversely, suppose that the π-scalar form B^k is given in the form (4.3), then, using Equation (4.1), one can show that the h-curvature tensor $\overset{\circ}{R}$ has the form

$$(\mathcal{P} \cdot \overset{\circ}{R})(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \varepsilon \, \mathfrak{A}_{\overline{X}, \overline{Y}} \left\{\hbar(\overline{X}, \overline{Z})\hbar(\overline{Y}, \overline{W})\right\}.$$

Therefore, from Definition 3.4, (M, L) is a Finsler manifold of H_p-scalar curvature. $\square$

Proposition 4.2. *A Finsler manifold of constant curvature k of dimension $n \geq 4$ is a Finsler manifold of H_p-constant curvature $\varepsilon = k$.*

Proof. If (M, L) is a Finsler manifold of constant curvature k of dimension $n \geq 4$, then the π-forms C^k and B^k vanish. Hence, by Theorem 3.1, the h-curvature tensor $\overset{\circ}{R}$ has the form

$$\overset{\circ}{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) \;\; = \;\; \mathfrak{A}_{\overline{X}, \overline{Y}} \{k\hbar(\overline{Y}, \overline{W})\{\ell(\overline{Z})\ell(\overline{X}) + \hbar(\overline{Z}, \overline{X})\} + kL^{-1}\hbar(\overline{X}, \overline{Z})\ell(\overline{W})\,\ell(\overline{Y})\}.$$

Applying the projection operator $\mathcal{P}$ on both sides of the above Equation, taking the fact that the angular metric tensor $\hbar$ is indicator and $\mathcal{P} \cdot \ell = 0$ (Remark 3.3) into account, we obtain

$$(\mathcal{P} \cdot \overset{\circ}{R})(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = k \, \mathfrak{A}_{\overline{X}, \overline{Y}} \left\{\hbar(\overline{X}, \overline{Z})\hbar(\overline{Y}, \overline{W})\right\}.$$

Therefore, (M, L) is a Finsler manifold of H_p-constant curvature $\varepsilon = k$. $\square$

Remark 4.1. The converse of the above proposition dose not hold, in general.

Proposition 4.3. *Let (M, L) be a Finsler manifold of scalar curvature k. If (M, L) is of H_p-scaler curvature ε, then we have*

(a) $3C^\varepsilon = 2C^k$,

(b) $3B^\varepsilon = 2B^k$,

(c) $3A^\varepsilon = 2A^k$,

where $A^\varepsilon := L\mathcal{P}\cdot \overset{v}{D}{}^{\circ} B^\varepsilon, \quad A^k := L\mathcal{P}\cdot \overset{v}{D}{}^{\circ} B^k.$ *Moreover, the above assertions are equivalent.*

Proof. **(a)** Suppose that (M, L) is a Finsler manifold of scalar curvature k and of H_p-scaler curvature ε. Then, by Theorem 4.1, we have

$$B^k(\overline{X}, \overline{Y}) = 3(\varepsilon - k)\hbar(\overline{X}, \overline{Y}).$$

Taking the vertical covariant derivative on both sides of the above equation, we get

$$(\overset{v}{D}{}^{\circ} B^k)(\overline{Z}, \overline{X}, \overline{Y}) = 3(\overset{v}{D}{}^{\circ} \varepsilon - \overset{v}{D}{}^{\circ} k)(\overline{Z})\hbar(\overline{X}, \overline{Y}) + 3(\varepsilon - k)(\overset{v}{D}{}^{\circ} \hbar)(\overline{Z}, \overline{X}, \overline{Y}).$$

Applying the projection operator $\mathcal{P}$ on both sides of the above equation and multiplying the resulting by L, we obtain

$$\begin{aligned}
L(\mathcal{P}\cdot \overset{v}{D}{}^{\circ} B^k)(\overline{Z}, \overline{X}, \overline{Y}) &= 3L(\mathcal{P}\cdot \overset{v}{D}{}^{\circ} \varepsilon - \mathcal{P}\cdot \overset{v}{D}{}^{\circ} k)(\overline{Z})(\mathcal{P}\cdot \hbar)(\overline{X}, \overline{Y}) \\
&\quad + 6L(\varepsilon - k)\{\mathcal{P}\cdot(T - L^{-1}\hbar \otimes \ell)\}(\overline{Z}, \overline{X}, \overline{Y}).
\end{aligned}$$

In view of Remark 3.3, the above equation reduces to

$$(4.4) \quad A^k(\overline{Z}, \overline{X}, \overline{Y}) = 3\{C^\varepsilon(\overline{Z}) - C^k(\overline{Z})\}\hbar(\overline{X}, \overline{Y}) + 6L(\varepsilon - k)T(\overline{Z}, \overline{X}, \overline{Y}).$$

On the other hand, using Lemma 3.1 of [22], we have

$$\mathfrak{A}_{\overline{X}, \overline{Y}}\left\{A^k(\overline{X}, \overline{Y}, \overline{Z}) + C^k(\overline{X})\hbar(\overline{Y}, \overline{Z})\right\} = 0,$$

Now, from which together with (4.4), it follows that

$$\mathfrak{A}_{\overline{X}, \overline{Y}}\left\{\{3C^\varepsilon(\overline{X}) - 2C^k(\overline{X})\}\phi(\overline{Y})\right\} = 0,$$

Taking the contracted trace with respect to $\overline{Y}$, we obtain

$$(n - 2)\{3C^\varepsilon(\overline{X}) - 2C^k(\overline{X})\} = 0.$$

Hence,

$$(4.5) \qquad\qquad 3C^\varepsilon = 2C^k \qquad\qquad (as\ n \geq 4).$$

(b) Taking the vertical covariant derivative on both sides of (4.5), we get

$$3\overset{v}{D}{}^{\circ} C^\varepsilon = 2\overset{v}{D}{}^{\circ} C^k.$$

Applying the projection operator $\mathcal{P}$ on both sides of the above equation and then multiplying by L, we obtain

$$3B^\varepsilon = 2B^k.$$

(c) The proof can be done in a similar manner as the proof of **(b)**.

Now, we prove the equivalences.

(a) $\Rightarrow$**(b)** $\Rightarrow$**(c):** The proof is obvious.

(c) $\Rightarrow$**(a):** Suppose that $3A^\varepsilon = 2A^k$ holds. Hence, using Lemma 3.1 of [22], we have

$$\mathfrak{A}_{\overline{X},\overline{Y}}\left\{\{3C^\varepsilon(\overline{X}) - 2C^k(\overline{X})\}\hbar(\overline{Y},\overline{Z})\right\} = 0$$

From which, the result follows. $\qquad\square$

Lemma 4.4. *[22] A Finsler manifold of scalar curvature k is of constant curvature if, and only if, $\mathcal{P} \cdot F = 0$, where F is the π-form defined by*

$$F(\overline{X},\overline{Y}) \quad := \quad \frac{1}{3}\left\{B(\overline{X},\overline{Y}) + 2C(\overline{X})\ell(\overline{Y})\right\}.$$

Theorem 4.5. *Let (M, L) be a Finsler manifold of scalar curvature k. If (M, L) is of H_p-scaler curvature ε, then, the following properties are equivalent:*

(a) $\varepsilon = k$.

(b) (M, L) *is of constant curvature k.*

(c) (M, L) *is of H_p-constant curvature ε.*

(d) $\mathcal{P} \cdot F = 0$

Proof. Assume that (M, L) is a Finsler manifold of scalar curvature k and of H_p-scaler curvature ε.

(a) $\Longleftrightarrow$**(b):** By making use of Theorem 4.1, we get

(4.6) $$B^k(\overline{X},\overline{Y}) = 3(\varepsilon - k)\hbar(\overline{X},\overline{Y}).$$

Now, if (M, L) is of constant curvature k, then

$$B^k(\overline{X},\overline{Y}) = 0$$

From which together with (4.6), it follows that $\varepsilon = k$.

Conversely, If (M, L) is a Finsler manifold of scalar curvature k and of H_p-scaler curvature ε with $\varepsilon = k$. Then due to (4.6), B^k vanishes. Hence, by Theorem 3.2, (M, L) is of constant curvature k.

(b) $\Longleftrightarrow$**(c):** Firstly, suppose that (M, L) is a Finsler manifold of scalar curvature k and of H_p-scaler curvature ε. If (M, L) is of constant curvature k, then, by Theorem 4.5, it follows that $\varepsilon = k$. Hence, (M, L) is of H_p-constant curvature.

Conversely, If (M, L) is a Finsler manifold of H_p-constant curvature ε. Then, C^ε vanishes. Hence, by Proposition 4.3, C^k vanishes. Consequently, by Theorem 3.2, (M, L) is of constant curvature k.

(b) $\Longleftrightarrow$**(d):** This is a direct consequence of Lemma 4.4. $\qquad\square$

Theorem 4.6. *Let (M, L) be a Finsler manifold of scalar curvature k. Then, (M, L) is of vanishing H_p-scaler curvature if, and only if, $\mathcal{P} \cdot N = 0$, where N is the π-form defined by*

$$N(\overline{X},\overline{Y}): \quad = \quad k\left\{g(\overline{X},\overline{Y}) + \ell(\overline{X})\ell(\overline{Y})\right\}$$
$$+\frac{1}{3}\left\{B(\overline{X},\overline{Y}) + 2\ell(\overline{X})C(\overline{Y}) + 2C(\overline{X})\ell(\overline{Y})\right\},$$

Proof. If (M, L) is a Finsler manifold of scalar curvature k. Then, by Theorem 3.1, we have

$$(4.7) \qquad (\mathcal{P} \cdot \overset{\circ}{R})(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \mathfrak{A}_{\overline{X}, \overline{Y}}\{\hbar(\overline{Y}, \overline{W})\{\frac{1}{3}B^k(\overline{Z}, \overline{X}) + k\hbar(\overline{Z}, \overline{X})\}\}.$$

On the other hand, by using the definition of N, we obtain

$$(\mathcal{P} \cdot N)(\overline{X}, \overline{Y}) = \frac{1}{3}B^k(\overline{Z}, \overline{X}) + k\hbar(\overline{Z}, \overline{X}).$$

From which together with (4.7), we get

$$(4.8) \qquad (\mathcal{P} \cdot \overset{\circ}{R})(\overline{X}, \overline{Y}, \overline{Z}) = \mathfrak{A}_{\overline{X}, \overline{Y}}\{\phi(\overline{Y})(\mathcal{P} \cdot N)(\overline{Z}, \overline{X})\}.$$

Now, if (M, L) is of vanishing H_p-scalar curvature, then , from (4.8), it follows that

$$\mathfrak{A}_{\overline{X}, \overline{Y}}\{\phi(\overline{Y})(\mathcal{P} \cdot N)(\overline{Z}, \overline{X})\}.$$

Taking the contracted trace with respect to $\overline{Y}$, the above relation reduces to

$$(n - 2)(\mathcal{P} \cdot N)(\overline{Z}, \overline{X}) = 0.$$

Consequently, as $n \geq 3$, $\mathcal{P} \cdot N$ vanishes.

Conversely, If (M, L) is a Finsler manifold of scalar curvature k such that $\mathcal{P} \cdot N = 0$. Hence, from (4.8), $\mathcal{P} \cdot \overset{\circ}{R}$ vanishes. Consequently, (M, L) is of vanishing H_p-scalar curvature. $\qquad \square$

5 Examples

In this section, we study various examples. Although the definition of the H_p-scalaer curvature manifolds requires that $n \geq 4$, we focus our attention to special cases of Example 5 with lower dimensions. This is only to show that the converse of some implications are not, generally, true. For instance, in Example 5, when $n = 2$ we show that the converse of Proposition 4.2 is not generally true. That is, not every Finsler manifold of H_p-constant curvature is of constant curvature. Also, in dimension 3, we show that not every Finsler manifold of H_p-scalar curvature is of scalar curvature.

Example 5.1. [4] The family of Riemannian metrics

$$L_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \ x \in \mathbb{B}^n(r_\mu), \ y \in T_x\mathbb{B}^n(r_\mu) \cong \mathbb{R}^n,$$

where $|.|$ and $\langle ., .\rangle$ are the standard Euclidean norm and inner product in $\mathbb{R}^n$, $r_\mu = 1/\sqrt{-\mu}$ if $\mu < 0$ and $r_\mu = \infty$ if $\mu \geq 0$. This class is of constant curvature μ. So it is also of H_p-constant curvature μ. As a verification, we have the following. The spray coefficients G^i of L_μ are given by

$$G^i = -\frac{\mu\langle x, y\rangle}{1 + \mu|x|^2}y^i.$$

From now on, (x^i, y^i) stands for the induced coordinates on the TM and we denote by ∂_i and $\dot{\partial}_i$ the partial differentiation with respect to the coordinates x^i and y^i respectively.

Firstly, we compute the coefficients G^i_j of the non linear connection. Since, $G^i_j = \dot{\partial}_j G^i$, we have

$$G^i_j = -\mu \frac{y^i x^\ell \delta_{\ell j} + \langle x, y \rangle \delta^i_j}{1 + \mu |x|^2}.$$

The the coefficients $G^i_{jk} = \dot{\partial}_k G^i_j$ of Berwald connection are given by

$$G^i_{jk} = -\mu \frac{x^\ell \delta_{\ell j} \delta^i_k + x^\ell \delta_{\ell k} \delta^i_j}{1 + \mu |x|^2}.$$

To calculate the components $\overset{\circ}{R}_{h\ jk}^{\ i}$ of the h-curvature of Berwald connection, we compute first the following

$$\partial_h G^i_{jk} = -\mu \frac{(1 + \mu |x|^2)(\delta^i_k \delta_{jh} + \delta_{kh} \delta^i_j) - 2\mu(x^\ell \delta_{\ell j} \delta^i_k + x^\ell \delta_{\ell k} \delta^i_j) x^m \delta_{mh}}{(1 + \mu |x|^2)^2}.$$

Now, using the formulae

$$\overset{\circ}{R}_{h\ jk}^{\ i} = \mathfrak{A}_{j,k} \{\delta_j G^i_k h + G^i_{mk} G^m_{hj}\}, \quad \delta_k = \partial_k - N^j_k \dot{\partial}_j,$$

we have

$$\overset{\circ}{R}_{h\ jk}^{\ i} = \mu(g_{hk} \delta^i_j - g_{hj} \delta^i_k).$$

Applying the projection operator on $\overset{\circ}{R}_{h\ jk}^{\ i}$, we get

$$\mathcal{P} \cdot \overset{\circ}{R}_{h\ jk}^{\ i} = \overset{\circ}{R}_{h\ jk}^{\ i} h^h_a h^j_b h^k_c h^d_i = \mu(h_{ac} h^d_b - g_{ab} h^d_c),$$

where h_{ij} is the angular metric. This gives $\varepsilon = \mu$.

Example 5.2. [12] The Funk metric on the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ is given by

$$L = \frac{\sqrt{|y|^2 - (|x|^2 |y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} + \frac{\langle x, y \rangle}{1 - |x|^2}.$$

The Funk metric has constant curvature $k = -\frac{1}{4}$ and hence has Hp-constant curvature $-\frac{1}{4}$.

Example 5.3. [5] The class

$$L_\phi = \frac{\langle a, y \rangle}{(1 + \langle a, x \rangle)^2} \phi(z^1, z^2, ..., z^n), \quad z^i = \frac{(1 + \langle a, x \rangle) y^i - \langle a, y \rangle x^i}{\langle a, y \rangle},$$

where $a = (a^1, a^2, ..., a^n)$ is a constant vector in $\mathbb{R}^n$ and ϕ is an arbitrary function in the sense that L_ϕ is a Finsler function. The spray coefficients are given by

$$G^i = -\frac{\langle a, y \rangle}{1 + \langle a, x \rangle} y^i.$$

The class L_ϕ is a class of Finsler metric of zero constant curvature and hence it is also of zero H_p-curvature. As in Example 1, straightforward calculations lead to the following.

$$G_j^i = -\frac{y^i a^\ell \delta_{\ell j} + \langle a, y \rangle \delta_j^i}{1 + \langle a, x \rangle}, \quad G_{jk}^i = -\mu \frac{a^\ell \delta_{\ell j} \delta_k^i + a^\ell \delta_{\ell k} \delta_j^i}{1 + \langle a, x \rangle},$$

$$\partial_h G_{jk}^i = \frac{(a^\ell \delta_{\ell j} \delta_k^i + a^\ell \delta_{\ell k} \delta_j^i) a^m \delta_{mh}}{(1 + \langle a, x \rangle)^2}, \quad \overset{\circ}{R}_{h\ jk}^{\ i} = 0.$$

Consequently, applying the projection operator on $\overset{\circ}{R}_{h\ jk}^{\ i}$ gives

$$\mathcal{P} \cdot \overset{\circ}{R}_{h\ jk}^{\ i} = 0,$$

which means that $\varepsilon = 0$.

Example 5.4. [3] Let $M = \mathbb{B}^n(1/\sqrt{|a|})$.

$$L = \frac{\sqrt{(1 - |a|^2 |x|^4)|y|^2 + (|x|^2 \langle a, y \rangle - 2\langle a, x \rangle \langle x, y \rangle)^2} - (|x|^2 \langle a, y \rangle - 2\langle a, x \rangle \langle x, y \rangle)}{1 - |a|^2 |x|^4}.$$

L is a Finsler metric of scalar curvature $k = 3\frac{\langle a, y \rangle}{F} + 3\langle a, x \rangle^2 - 2|a|^2 |x|^2$.

Now to decide whether the space (M, L) is of H_p-scalar curvature, we have to calculate the tensor $k_{ij} := F\dot{\partial}_j(F\dot{\partial}_i k)$.

$$\dot{\partial}_i k = 3\frac{F a_i - \langle a, y \rangle \ell_i}{F^2}.$$

Hence,

$$k_{ij} = -3a_i \ell_j - 3\frac{\langle a, y \rangle}{F^2} h_{ij}.$$

Applying the projection operator on k_{ij}, we have

$$B_{ij} = \mathcal{P} \cdot k_{ij} = k_{ab} h_i^a h_j^b = -3\frac{\langle a, y \rangle}{F} h_{ij}.$$

Therefore, by using Theorem 4.1, the space (M, L) is of Hp-scalar curvature. Here the Hp-scalar curvature ε is given by

$$\varepsilon = \frac{2\langle a, y \rangle}{F} + 3\langle a, x \rangle^2 - 2|a|^2 |x|^2.$$

Example 5.5. Let $M = \mathbb{R}^n$ and

$$L = f(x^1)|y|,$$

where $f(x^1)$ is an arbitrary smooth positive function on $\mathbb{R}$ and $|y|$ is the standard norm on $\mathbb{R}^n$. The metric tensor is given by

$$g_{ij} = f(x^1)^2 \delta_{ij}, \quad g^{ij} = f(x^1)^{-2} \delta^{ij}, \quad h_{ij} = f(x^1)^2(|y|\delta_{ij} - y_i y_j).$$

The Berwald connection is given by

$$G_{ij}^h = \phi(x^1)(\delta_{1i}\delta_j^h + \delta_{1j}\delta_i^h - \delta_{ij}\delta_1^h), \quad \phi(x^1) := \frac{f'(x^1)}{f(x^1)}, \quad f'(x^1) := \frac{df(x^1)}{dx^1}.$$

By differentiating G_{ij}^h in the following sense

$$\partial_k G_{ij}^h = \phi'(x^1)\delta_{1k}(\delta_{1i}\delta_j^h + \delta_{1j}\delta_i^h - \delta_{ij}\delta_1^h), \quad \phi'(x^1) := \frac{d\phi(x^1)}{dx^1},$$

the h-curvature of Berwald connection is given by

$$\mathring{R}_{i\ jk}^{\ h} = \mathfrak{A}_{j,k}\{\phi'(\delta_{1i}\delta_{1j}\delta_k^h + \delta_{ij}\delta_{1k}\delta_1^h) + \phi^2(\delta_k^h\delta_{ij} + \delta_{1i}\delta_{1k}\delta_j^h + \delta_{1j}\delta_{ik}\delta_1^h)\}.$$

In dimension 3, by using Maple program and NF-package [21], one can see that the space (M, L) is not of scalar curvature. But it satisfies the condition of the Hp-scalar curvature. Namely,

$$\mathcal{P} \cdot \mathring{R}_{i\ jk}^{\ h} = \mathring{R}_{b\ cd}^{\ a} = -\frac{1}{L^2}(\phi'((y^2)^2 + (y^3)^2) + \phi(y^1)^2)(h_c^a h_{bd} - h_d^a h_{bc}).$$

Hence, $\varepsilon = -\frac{1}{L^2}(\phi'(((y^2)^2 + (y^3)^2) + \phi(y^1)^2)$.

In dimension 2, by using Maple program and NF-package, one can see that the space (M, L) is of scalar curvature $k = \frac{f'^2 - ff''}{f^4}$. But applying the projection operator on the h-curvature gives

$$\mathcal{P} \cdot \mathring{R}_{i\ jk}^{\ h} = 0.$$

Hence, $\varepsilon = 0$. This case shows that the converse of Proposition 4.2 is not generally true.

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