On \( f \)-injective modules

By

MAHER ZAYED

Abstract. In this paper, the notions of \( f \)-injective and \( f^* \)-injective modules are introduced. Elementary properties of these modules are given. For instance, a ring \( R \) is coherent if any ultraproduct of \( f \)-injective modules is absolutely pure. We prove that the class \( \sum^{*} \) of \( f^* \)-injective modules is closed under ultraproducts. On the other hand, \( \sum^{*} \) is not axiomatisable. For coherent rings \( R \), \( \sum^{*} \) is axiomatisable if every \( \chi_0 \)-injective module is \( f^* \)-injective. Further, it is shown that the class \( \sum \) of \( f \)-injective modules is axiomatisable if \( R \) is coherent and every \( \chi_0 \)-injective module is \( f \)-injective. Finally, an \( f \)-injective module \( H \), such that every module embeds in an ultrapower of \( H \), is given.

1. Introduction. In [3], Eklof and Sabbagh introduced the notion of \( \alpha \)-injective module. For a cardinal \( \alpha \geq 2 \), a module \( X \) over a ring \( R \) is \( \alpha \)-injective if for every ideal \( I \) having a generating subset of less than \( \alpha \) elements, any homomorphism of \( I \) into \( X \) can be extended to a homomorphism of \( R \) into \( X \). In this paper, the notions of \( f \)-injective and \( f^* \)-injective modules are introduced. An \( R \)-module \( X \) is said to be \( f \)-injective (resp. \( f^* \)-injective) if given any monomorphism \( F \to Y \), where \( F \) is a finitely generated (resp. finitely presented) module, any homomorphism \( F \to X \) can be extended to a homomorphism \( Y \to X \).

Note that every \( f \)-injective is \( \chi_0 \)-injective and the converse is not generally true (Remark 3.4).

Elementary properties of these modules are given. For instance, a ring \( R \) is coherent if and only if any ultraproduct of \( f \)-injective modules is absolutely pure. We prove that the class \( \sum^{*} \) of \( f^* \)-injective modules is closed under ultraproducts. On the other hand, \( \sum^{*} \) is not axiomatisable. For coherent rings \( R \), \( \sum^{*} \) is axiomatisable if and only if every \( \chi_0 \)-injective module is \( f^* \)-injective. Further, it is shown that the class \( \sum \) of \( f \)-injective modules is axiomatisable if and only if \( R \) is coherent and every \( \chi_0 \)-injective module is \( f \)-injective. Finally, an \( f \)-injective module \( H \), such that every module embeds in an ultrapower of \( H \), is given.

2. Notation and preliminary results. Throughout this paper, \( R \) is an associative ring with identity and all modules are left unitary \( R \)-modules. The class of finitely generated \( R \)-modules is denoted by \( f \). The subclass of \( f \) whose objects are the finitely presented modules in \( f \) is denoted by \( f^* \). An \( R \)-module \( X \) is said to be \( f \)-injective (resp. \( f^* \)-injective) if for every monomorphism \( f : F \to Y \), \( F \in f \) (resp. \( F \in f^* \)), any homomorphism \( g : F \to X \) can be extended to a homomorphism \( h : Y \to X \); that is \( g = h \circ f \).

Mathematics Subject Classification (2000): 16D70, 16D80, 12L10, 03C60.
Proposition 2.1. (a) A direct product $\prod_{\alpha \in A} X_\alpha$ of modules is $f$-injective if and only if each $X_\alpha$ is $f$-injective.

(b) If $X_0 \subset X_1 \subset \ldots \subset X_\beta \subset \ldots$, $\beta < \alpha$ is a chain of $f$-injective modules, where $\alpha$ is an ordinal, then the union of the chain is $f$-injective.

(c) Any direct sum of $f$-injective modules is $f$-injective.

(d) Every module has a maximal $f$-injective submodule.

(e) Every finitely generated (resp. finitely presented) $f$-injective (resp. $f^*$-injective) module is injective.

Proof. Easy.

Corollary 2.2. A ring $R$ is left noetherian if and only if every $f$-injective $R$-module is injective.

Proof. The ‘only if’ part follows from Baer’s criterion of injectivity. The ‘if’ part follows from Proposition 2.1(c) and [1, Prop. 18.13].

Corollary 2.3. A ring $R$ is semi-simple artinian if and only if every $R$-module is $f$-injective.

Proof. Apply Proposition 2.1(e) and [8, Theorem].

3. Ultraproducts of $f^*$-injective modules. Let $I$ be a nonempty set, $(X_i)_{i \in I}$ be a family of $R$-modules and $u$ be an ultrafilter on $I$. The ultraproduct of this family with respect to $u$ is denoted by $\Pi_i \in I X_i$. If $X_i = X$ for all $i \in I$, the ultraproduct is denoted by $X^I/u$ and is called the ultrapower of $X$. For the basic concepts of model theory and the main properties of ultraproducts of algebraic structures we refer to [2] and [6]. Let $X$ and $Y$ be two modules over $R$. $X$ and $Y$ are called elementarily equivalent (notation: $X \equiv Y$) if $X$ and $Y$ satisfy the same first order sentences in the language of modules over $R$. A class $K$ of $R$-modules is called axiomatisable if there exists a family of first order sentences in the language of modules over $R$ such that $K$ consists exactly of the modules satisfying these first order sentences. Let $\Sigma$ (resp. $\Sigma^*$) be the class of all $f$-injective (resp. $f^*$-injective) $R$-modules. If $\Gamma$ denotes the class of injective $R$-modules, then $\Gamma \subseteq \Sigma \subseteq \Sigma^*$. Note that if $R$ is left coherent, then every $f^*$-injective $R$-module is $\chi_0$-injective.

Proposition 3.1. $\Sigma^*$ is closed under ultraproducts.

Proof. Let $(X_i)_{i \in I}$ be a family of $f^*$-injective modules and $u$ be a non-principal ultrafilter on $I$. Let $F \in f^*$ and consider the following diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & F \\
& & \downarrow \Phi \\
& & \Pi_i X_i
\end{array}
$$

Since $F$ is finitely presented, so there exist a set $\Omega \in u$ and a homomorphism $\lambda : F \to \Pi_i X_i$, such that $g = \Phi \circ \lambda$, where $\Phi : \Pi_i X_i \to \Pi_i X_i$ is the canonical homomorphism [5].

Note that $\Pi_i X_i \in \Sigma^*$, so there exists $h : Y \to \Pi_i X_i$ such that $h \circ f = \lambda$.

Now, if $\gamma = \Phi \circ h : Y \to \Pi_i X_i$, then $\gamma \circ f = \Phi \circ h \circ f = \Phi \circ \lambda = g$. This means that $\Pi_i X_i$ belongs to $\Sigma^*$. 

Corollary 3.2. Any ultraproduct of $f$-injective (resp. injective) $R$-modules is $f^*$-injective.

Corollary 3.3. $\sum^*$ is elementarily closed if and only if $\sum^0$ is closed under elementary descent.

Proof. The ‘only if’ part is obvious. The ‘if’ part follows from Frayne’s Lemma [2, Ch. 8, Lemma 1.1] and Proposition 3.1.

Remark 3.4. Let $V$ be an infinite dimensional vector space over a division ring $D$ and $R = \text{End}(V_D)$. The ring $R$ is von Neumann regular. Further $R$ is not left self-injective. In fact $R$ has a primitive idempotent $e$ such that $M = Re$ is not injective [1, Ex. 18.4].

Observe that $M$ is $\chi_0$-injective and by Proposition 2.1(e), $M \not\in \sum^*$. Let $E(M)$ be the pure-injective envelope of $M$. Since $R$ is regular, $E(M)$ is injective and so $E(M) \in \sum^*$. Note that $M \equiv E(M)$ [9]. Hence, in general, the class $\sum^*$ is not elementarily closed. It follows from [2, Ch. 7. Theorem 3.4], that $\sum^*$ is not an axiomatisable class.

We do not know for what rings the $f^*$-injective modules form an axiomatisable class. However, for coherent rings, one easily obtains the following result.

Proposition 3.5. Let $R$ be a left coherent ring and $\sum^0$ be the class of all $\chi_0$-injective $R$-modules. Then $\sum^*$ is axiomatisable if and only if $\sum^0 = \sum^*$.

Proof. Suppose that $\sum^*$ is axiomatisable and $X \in \sum^0$. By Lemma 3.17 of [3], $X$ is an elementary submodule of an injective module $I$.

Since $I \in \sum^*$, then $X \in \sum^*$. The converse results from [3, Theorem 3.16].

Corollary 3.6. For a regular ring $R$, $\sum^*$ is axiomatisable if and only if every $R$-module is $f^*$-injective.

4. Ultraproducts of $f$-injective modules. In this section, we show that, if $\sum$ is axiomatisable, then $R$ is left coherent. It follows from the preceding remark that the converse is not generally true. However, a ‘converse’ of this result will be proved.

Proposition 4.1. The following assertions are equivalent:

(i) $\sum$ is closed under ultraproducts.
(ii) $\sum$ is closed under ultrapowers.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. To show that (ii) $\Rightarrow$ (i), let $(X_i)_{i \in I}$ be a family of $f$-injective modules and $u$ be a non-principal ultrafilter over $I$. By [3, Remark p. 261], the $R$-module $\Pi_u X_i$ is a direct summand of an ultra-power of the direct product of the family $(X_i)_{i \in I}$. The result follows from Proposition 2.1.

We recall that an $R$-module $M$ is called absolutely pure (or $f$-p-injective) if each short exact sequence $0 \rightarrow M \rightarrow A \rightarrow B \rightarrow 0$ of $R$-modules is pure-exact. It is an equivalent assertion that every $R$-linear map $f : U \rightarrow M$, where $U$ is a finitely generated submodule of a finitely generated free module $F$, admits an extension to $F$. Of course, every $f$-injective $R$-module is absolutely pure.
Proposition 4.2. The following conditions are equivalent:

(i) $R$ is left coherent.
(ii) Any ultraprocess of $f$-injective $R$-modules is absolutely pure.
(iii) Any ultrapower of $f$-injective $R$-module is absolutely pure.

Proof. (i) $\Rightarrow$ (ii) follows from [9, Theorem 2], and (ii) $\Rightarrow$ (iii) is obvious. So, it remains to show (iii) $\Rightarrow$ (i). Let $(X_i)_{i \in I}$ be a family of injective $R$-modules and $u$ be a non-principal ultrafilter on $I$. The direct product $X = \Pi_{i \in I} X_i$ is $f$-injective. Under the hypothesis (iii), any ultrapower of $X$ is absolutely pure. Note that any direct summand of an absolutely pure module is absolutely pure [7]. So, the ultraproduct $\Pi_u X_i$ (which is a summand of an ultrapower of $X$) is absolutely pure. Now, $R$ is left coherent follows from Theorem 2 of [9].

Corollary 4.3. We consider the following assertions:

(i) $\Sigma$ is axiomatisable.
(ii) $\Sigma$ is elementarily closed.
(iii) $\Sigma$ is closed under ultraproducts.
(iv) $R$ is left coherent.

Then (i) $\iff$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

The proof of Proposition 3.5 can be easily modified to yield the following:

Proposition 4.4. For a ring $R$, the class $\Sigma$ is axiomatisable if and only if $R$ is left coherent and $\Sigma_0 = \Sigma$.

Corollary 4.5. For a regular ring $R$, $\Sigma$ is axiomatisable if and only if $R$ is semisimple artinian.

Proposition 4.6. For any ring $R$, there is an $f$-injective $R$-module $H$, such that every module embeds in an ultrapower of $H$.

Proof. Let $H = \oplus \{ I(M) : M \text{ is finitely generated} \}$, where $I(M)$ is the injective envelope of $M$. Let $X$ be any module and $\{ B_j : j \in J \}$ be the set of all finitely generated submodules of $X$. For each $j \in J$, $B_j$ is embedded in $I(B_j)$. So, there exists an embedding $f_j : B_j \rightarrow H$. For each $j \in J$. By [4, Theorem 6.1], there is an ultrafilter $u$ on $J$ and an embedding of $X$ into the ultrapower $H^J/u$ of $H$, observe that $H$ is $f$-injective.

Acknowledgement. The author would like to thank the referee for his useful suggestions.

References

On $f$-injective modules


Anschrift des Autors:
Maher Zayed
Department of Mathematics
Faculty of Science, University of Banha
Banha 13518, Egypt

Eingegangen am 12. 7. 2000