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On f-injective modules

By

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Abstract. In this paper, the notions of f-injective and f^*-injective modules are indroduced. Elementary properties of these modules are given. For instance, a ring R is coherent iff any ultraproduct of f-injective modules is absolutaly pure. We prove that the class $\sum^*$ of f^*-injective modules is closed under ultraproducts. On the other hand, $\sum^*$ is not axiomatisable. For coherent rings R, $\sum^*$ is axiomatisable iff every χ_0 -injective module is f^*-injective. Further, it is shown that the class $\sum$ of f-injective modules is axiomatisable iff R is coherent and every χ_0-injective module is f-injective. Finally, an f-injective module H, such that every module embeds in an ultraprower of H, is given.

1. Introduction. In [3], Eklof and Sabbagh introduced the notion of α-injective module. For a cardinal $\alpha \geqq 2$, a module X over a ring R is α-injective if for every ideal I having a generating subset of less than α elements, any homomorphism of I into X can be extended to a homomorphism of R into X. In this paper, the notions of f-injective and f^*-injective modules are introduced. An R-module X is said to be f*-injective* (resp. f^**-injective*) if given any monomorphism $F \to Y$, where F is a finitely generated (resp. finitely presented) module, any homomorphism $F \to X$ can be extended to a homomorphism $Y \to X$.

Note that every f-injective is χ_0-injective and the converse is not generally true (Remark 3.4).

Elementary properties of these modules are given. For instance, a ring R is coherent if and only if any ultraproduct of f-injective modules is absolutely pure. We prove that the class $\sum^*$ of f^*-injective modules is closed under ultraproducts. On the other hand, $\sum^*$ is not axiomatisable. For coherent rings R, $\sum^*$ is axiomatisable if and only if every χ_0-injective module is f^*-injective. Further, it is shown that the class $\sum$ of f-injective modules is axiomatisable if and only if R is coherent and every χ_0-injective module is f-injective. Finally, an f-injective module H, such that every module embeds in an ultrapower of H, is given.

2. Notation and preliminary results. Throughout this paper, R is an associative ring with identity and all modules are left unitary R-modules. The class of finitely generated R-modules is denoted by f. The subclass of f whose objects are the finitely presented modules in f is denoted by f^*. An R-module X is said to be f*-injective* (resp. f^**-injective*) if for every monomorphism $f : F \to Y$, $F \in f$, (resp. $F \in f^*$), any homomorphism $g : F \to X$ can be extended to a homomorphism $h : Y \to X$; that is $g = h \circ f$.

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Proposition 2.1. (a) *A direct product* $\prod_{\alpha\in A} X_\alpha$ *of modules is* f*-injective if and only if each* X_α *is* f*-injective.*
(b) *If* $X_0 \subset X_1 \subset .... \subset X_\beta \subset ..., \beta \prec \alpha$ *is a chain of* f*-injective modules, where* α *is an ordinal, then the union of the chain is* f*-injective.*
(c) *Any direct sum of* f*-injective modules is* f*-injective.*
(d) *Every module has a maximal* f*-injective submodule.*
(e) *Every finitely generated (resp. finitely presented)* f*-injective (resp.* f^**-injective) module is injective.*

Proof. Easy.

Corollary 2.2. *A ring* R *is left noetherian if and only if every* f*-injective* R*-module is injective.*

Proof. The 'only if' part follows from Baer's criterion of injectivity. The 'if' part follows from Proposition 2.1(c) and [1, Prop. 18.13].

Corollary 2.3. *A ring* R *is semi-simple artinian if and only if every* R*-module is* f*-injective.*

Proof. Apply Proposition 2.1(e) and [8, Theorem].

3. Ultraproducts of f^***-injective modules.** Let I be a nonempty set, $(X_i)_{i\in I}$ be a family of R-modules and u be an ultrafilter on I. The ultraproduct of this family with respect to u is denoted by $\Pi_u X_i$. If $X_i = X$ for all $i \in I$, the ultraproduct is denoted by X^I/u and is called the ultrapower of X. For the basic concepts of model theory and the main properties of ultraproducts of algebraic structures we refer to [2] and [6]. Let X and Y be two modules over R. X and Y are called elementarily equivalent (notation: $X \equiv Y$) if X and Y satisfy the same first order sentences in the language of modules over R. A class K of R-modules is called axiomatisable if there exists a family of first order sentences in the language of modules over R such that K consists exactly of the modules satisfying these first order sentence. Let $\sum$ (resp. $\sum^*$) be the class of all f-injective (resp. f^*-injective) R-modules. If Γ denotes the class of injective R-modules, then $\Gamma \subseteqq \sum \subseteqq \sum^*$. Note that if R is left coherent, then every f^*-injective R-module is χ_0-injective .

Proposition 3.1. $\sum^*$ *is closed under ultraproducts.*

Proof. Let $(X_i)_{i\in I}$ be a family of f^*-injective modules and u be a non-principal ultrafilter on I. Let $F \in f^*$ and consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & F & \xrightarrow{f} & Y \\ & & {\scriptstyle g}\big\downarrow & & \\ & & \Pi_u X_i & & \end{array}$$

Since F is finitely presented, so there exist a set $\Omega \in u$ and a homomorphism $\lambda : F \to \prod_{i\in\Omega} X_i$, such that $g = \Phi \circ \lambda$ where $\Phi : \prod_{i\in\Omega} X_i \to \Pi_u X_i$ is the canonical homomorphism [5].

Note that $\prod_{i\in\Omega} X_i \in \sum^*$, so there exists $h : Y \to \prod_{i\in\Omega} X_i$ such that $h \circ f = \lambda$.

Now, if $\gamma = \Phi \circ h : Y \to \Pi_u X_i$, then $\gamma \circ f = \Phi \circ h \circ f = \Phi \circ \lambda = g$. This means that $\Pi_u X_i$ belongs to $\sum^*$.

Corollary 3.2. *Any ultraproduct of f-injective (resp. injective) R-modules is f^*-injective.*

Corollary 3.3. *$\sum^*$ is elementarily closed if and only if $\sum^*$ is closed under elementary descent.*

P r o o f. The 'only if' part is obvious. The 'if' part follows from Frayne's Lemma [2, Ch. 8, Lemma 1.1] and Proposition 3.1.

R e m a r k 3.4. Let V be an infinite dimensional vector space over a division ring D and $R = End(V_D)$. The ring R is von Neumann regular. Further R is not left self-injective. In fact R has a primitive idempotent e such that $M = Re$ is not injective [1, Ex. 18.4].

Observe that M is χ_0-injective and by Proposition 2.1(e), $M \notin \sum^*$.

Let $E(M)$ be the pure-injective envelope of M. Since R is regular, $E(M)$ is injective and so $E(M) \in \sum^*$. Note that $M \equiv E(M)$ [9]. Hence, in general, the class $\sum^*$ is not elementarily closed. It follows from [2, Ch. 7. Theorem 3.4], that $\sum^*$ is not an axiomatisable class.

We do not know for what rings the f^*-injective modules form an axiomatisable class. However, for coherent rings, one easily obtains the following result.

Proposition 3.5. *Let R be a left coherent ring and $\sum_0$ be the class of all χ_0-injective R-modules. Then $\sum^*$ is axiomatisable if and only if $\sum_0 = \sum^*$.*

P r o o f. Suppose that $\sum^*$ is axiomatisable and $X \in \sum_0$. By Lemma 3.17 of [3], X is an elementary submodule of an injective module I.

Since $I \in \sum^*$, then $X \in \sum^*$. The converse results from [3, Theorem 3.16].

Corollary 3.6. *For a regular ring R, $\sum^*$ is axiomatisable if and only if every R-module is f^*-injective.*

4. Ultraproducts of f-injective modules. In this section, we show that, if $\sum$ is axiomatisable, then R is left coherent. It follows from the preceding remark that the converse is not generally true. However, a"converse' of this result will be proved.

Proposition 4.1. *The following assertions are equivalent:*

(i) *$\sum$ is closed under ultraproducts.*
(ii) *$\sum$ is closed under ultrapowers.*

P r o o f. The implication (i) $\Rightarrow$ (ii) is obvious. To show that (ii) $\Rightarrow$ (i), let $(X_i)_{i \in I}$ be a family of f-injective modules and u be a non-principal ultrafilter over I. By [3, Remark p. 261], the R-module $\Pi_u X_i$ is a direct summand of an ultra-power of the direct product of the family $(X_i)_{i \in I}$. The result follows from Proposition 2.1.

We recall that an R-module M is called absolutely pure (or f p-injective) if each short exact sequence $0 \to M \to A \to B \to 0$ of R-modules is pure-exact. It is an equivalent assertion that every R-linear map $f : U \to M$, where U is a finitely generated submodule of a finitely generated free module F, admits an extension to F. Of course, every f-injective R-module is absolutely pure.

Proposition 4.2. *The following conditions are equivalent:*

(i) *R is left coherent.*
(ii) *Any ultraproduct of f-injective R-modules is absolutely pure.*
(iii) *Any ultrapower of f-injective R-module is absolutely pure.*

Proof. (i) $\Rightarrow$ (ii) follows from [9, Theorem 2]. and (ii) $\Rightarrow$ (iii) is obvious. So, it remains to show (iii) $\Rightarrow$ (i). Let $(X_i)_{i\in I}$ be a family of injective R-modules and u be a non-principal ultrafilter on I. The direct product $X = \Pi_{i\in I} X_i$ is f-injective. Under the hypothesis (iii), any ultrapower of X is absolutely pure. Note that any direct summand of an absolutely pure module is absolutely pure [7]. So, the ultraproduct $\Pi_u X_i$ (which is a summand of an ultrapower of X) is absolutely pure. Now, R is left coherent follows from Theorem 2 of [9].

Corollary 4.3. *We consider the following assertions:*

(i) $\sum$ *is axiomatisable.*
(ii) $\sum$ *is elementarily closed.*
(iii) $\sum$ *is closed under ultraproducts.*
(iv) *R is left coherent.*

Then (i) $\Longleftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

The proof of Proposition 3.5 can be easily modified to yield the following:

Proposition 4.4. *For a ring R, the class* $\sum$ *is axiomatisable if and only if R is left coherent and* $\sum_0 = \sum$.

Corollary 4.5. *For a regular ring R,* $\sum$ *is axiomatisable if and only if R is semisimple artinian.*

Proposition 4.6. *For any ring R, there is an f-injective R-module H, such that every module embeds in an ultrapower of H.*

Proof. Let $H = \oplus\{I(M) : M$ is finitely generated$\}$, where $I(M)$ is the injective envelope of M. Let X be any module and $\{B_j : j \in J\}$ be the set of all finitely generated submodules of X. For each $j \in J$. B_j is embedded in $I(B_j)$. So, there exists an embedding $f_j : B_j \to H$, For each $j \in J$. By [4, Theorem 6.1], there is an ultrafilter u on J and an embedding of X into the ultrapower H^J/u of H. observe that H is f-injective.

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