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Sub-Harmonic Solutions of Even Order $(\frac{1}{2}, \frac{1}{4})$, To A Weakly Non-Linear Second Order Differential Equation Governed the Motion (MEMS)

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Abstract: *Sub-harmonic periodic solutions of even order* $(\frac{1}{2}, \frac{1}{4})$ *to a weakly non-linear second order differential equations which governed the motion of a micro-electro mechanical system (MEMS) (Bandpass Filter) are investigated analytically. The method of multiple scales is used to determine the modulation equations in the amplitude and the phase, steady state solutions. The frequency-response equation and stability analysis of the steady state solutions are obtained. Numerical study of the frequency-response equations and stability equations are given for different values of the parameters. Results are plotted in group of Figures, on which solid (dashed) curves are stable (unstable) solutions. Finally discussion and conclusion are given.*

Key Words: *MEMS, Weakly non-linear differential equation, Multiple scales method, Stability and Parametric Excitation*.

1 Introduction

The advent of micro-bifurcation technologies in the last couple of decades has led to the birth of an exciting and revolutionary field called micro-dynamical systems. Micro systems are literally very small systems or systems made of very small components. For micro-electro-mechanical systems(MEMS) micro-establishes a dimensional scale electro-suggests either electricity or electronic (or both) and mechanical suggests moving parts of some kind[1].

Lid and Wang[2] studied structural dynamics of microsystems, current state of research and future directions. Mathematical study of this type of dynamical systems leads to a non-linear second order differential equations or a set of coupled non-linear second order differential equations. Periodic solutions of this type of differential equations (the response of its dynamical systems) is the object of many works. Elnaggar et al.[3- 6] studied different kinds of periodic solutions (harmonic, sub-harmonic and super-harmonic solutions) of a weakly non-linear second order differential equation. L-Cveticanin et al.[7] studied the periodic solution of the generalized Rayleigh equation. Younis and Nayfeh [8] used the method of multiple scales to study the response of an electrostatically actuated resonator to resonance excitation. Abd El-Rahman and Nayfeh [9, 10] studied a super-harmonic resonance excitation of order one-half. Zhang and Meng [11] analyzed the nonlinear dynamics of the electrostatically actuated resonant sensors under parametric excitation. Elnaggar et al.[12] studied harmonic and sub-harmonic resonance of micro-electro mechanical system(MEMS) subjected to a weakly non-linear parametric and external excitation. Zavodney and Nayfeh [13] studied the response of a single-degree-of-freedom system with quadratic and cubic non-linearities to a fundamental parametric resonance. Elnaggar and Alhanadwah [14] studied parametric excitation of subharmonic oscillations. Elnaggar and El-Bassiouny [15, 16] studied the response of self-excited two-degree and three-degree-of-freedom systems to multi-frequency excitations. El-Dib [17] used the method of multiple scales to determine a third-order solution for a cubic non-linear Mathieu equation. El-Bassiouny and Eissa [18] investigated analytical and numerical solutions of single-degree-of-freedom with quadratic and cubic nonlinearities to a harmonic resonance. El-Dib [19] analyzed a theoretical analysis of the parametric harmonic response of two resonant modes based on a cubic non-linear system. El-Bassiouny et al.[20] investigated two-to-one internal resonances in nonlinear two-degree-of-freedom system with parametric and external excitations. Rhoads et al.[21] studied the parametric resonance of micro-electro mechanical system, which represented mathematically by a weakly non-linear second order differential equation in which the non-linearity is modeled by a cubic function of displacement. El-Bassiouny [22] studied an approach for implementing an active nonlinear vibration absorber. The strategy exploits the saturation phenomenon that is exhibited by multi-degree-of-freedom systems with cubic nonlinearities possessing one-to-one internal resonance. The proposed technique consists of introducing a second-order controller and coupling it to the plant through a sensor and an actuator where both the feedback and control signals are cubic. El-Bassiouny [23] studied the dynamical stability and complicated motions of a vessel in regular sea are investigated when the frequancy in the pitch is nearly twice the frequency in the roll. El-Bassiouny[24] investigated vibration and chaos control of nonlinear torsional vibrating systems. Zhang et al. [25] studied the dynamics of nonlinear coupled electrostatic micro mechanical resonators under two frequency parametric and external excitations. Elnaggar et al.[26] used the method of multiple scales to investigated the saddle node bifurcation control for an odd non-linearity problem. Elnaggar et al.[27] analyzed the perturbation analysis of an electrostatic Micro-Electro-Mechanical System(MEMS) subjected to external and non-linear parametric excitations. Harmonic, sub-harmonic and super-harmonic resonance of weakly non-linear dynamical system subjected to external excitation, parametric excitation or both are investigated by Elnaggar et al.[28]. Kacem et al.[29- 31] studied respectively nonlinear multi-physics models including both mechanical and electrostatic nonlinearities and the fringing field effect. Jeffrey F. Rhoads et al.[32] studied the tunable micro electro mechanical filters that exploit parametric resonance in which the elastic restoring force is modeled by a cubic function. A more comprehensive review up to 2010 about the work on the non-linear dynamics of micro resonators is presented in literature.

In this paper an analytical study (perturbation analysis) for a weakly non-linear second order differential equation, which governed the motion of micro electro-mechanical system(MEMS)(Bandpass Filter)[32] in which the elastic restoring force is modeled by an odd non-linear function. The object of the study is to determine different types of periodic solutions sub-harmonic of order $(\frac{1}{2},\frac{1}{4})$ and its stability. Perturbation method (multiple scales method) [33-36] is used to determine, the modulation equations in the amplitude and the phase. Steady-state solutions, the frequency-response equations and the stability analysis are given. Finally numerical study for frequency-response and the stability equations are carried. The results are plotted in group of Figures in which solid(dashed) lines means stable(unstable) solutions. Discussion and conclusions are given.

2 Formulation of the problem and perturbation analysis

The oscillation motion the Micro-Electro Mechanical Systems (MEMS)(Bandpass Filter)[32], is governed by the following weakly non-linear second order differential equation

$$u'' + 2\varepsilon\zeta u' + \omega_o^2 u + \varepsilon(F_1 u + F_3 u^3 + F_5 u^5 + F_7 u^7) + \varepsilon(H_1 u + H_2 u^3)\cos(\Omega\tau) = 0 \qquad \ldots (1)$$

Equation (1) represent Modified Duffing's equation subjected to weakly non-linear parametric excitation, where the dots indicate differentiation with respect to t, ε is a small parameter $\varepsilon = 1$, ζ is the coefficient of viscous damping, ω_o is the linear natural frequency, Ω is frequency of the external excitation, F_1, F_3, F_5 and F_7 are the coefficients of the non-linear terms, H_1 and H_2 are the coefficients of linear and nonlinear parametric excitations respectively. To determine a first-order uniform expansion of the solutions of Eq.(1), one can use the method of multiple scales [33-36]. Let

$$u(t;\varepsilon) = u_o(T_o, T_1) + \varepsilon u_1(T_o, T_1) + O(\varepsilon^2), \quad T_n = \varepsilon^n t \qquad \dots (2)$$

where $T_o = t$ is the first scale associated with changes occurring at the frequencies ω_o and Ω, and $T_1 = \varepsilon t$ is a slow scale associated with modulations in the amplitude.

$$\frac{d}{dt} = D_o + \varepsilon D_1 + \dots\dots.., \frac{d^2}{dt^2} = D_o^2 + 2\varepsilon D_o D_1 + \dots\dots \qquad \dots (3)$$

where $D_n = \dfrac{\partial}{\partial T_n}$. Substituting Eqs.(2) and (3) into Eq.(1) and equating terms with the same power of on both sides, we obtain a system of linear partial differential equations

$$O(1): D_o^2 u_o + \omega_o^2 u_o = 0 \qquad \dots (4)$$

$$\begin{aligned} O(\varepsilon): D_o^2 u_1 + \omega_o^2 u_1 = \; & -2D_o D_1 u_o - 2\zeta D_o u_o - F_1 u_o - F_3 u_o^3 - F_5 u_o^5 \\ & - F_7 u_o^7 - (H_1 u_o + H_2 u_o^3)\cos(\Omega T_o) \end{aligned} \qquad \dots (5)$$

The solution of Eq.(4) can be expressed by

$$u_o(T_o, T_1) = A(T_1) e^{i\omega_o T_o} + \bar{A}(T_1) e^{-i\omega_o T_o} \qquad \dots (6)$$

where A is the amplitude of the solution and is a function of T_1 and $\bar{A}$ is the complex conjugate of A, substitute Eq.(6)into Eq.(5), we get

$$\begin{aligned} D_o^2 u_1 + \omega_o^2 u_1 = \; & -(2i\omega_o A' + 2i\omega_o \zeta A + F_1 A + 3F_3 A^2 \bar{A} + 10F_5 A^3 \bar{A}^2 + 35F_7 A^4 \bar{A}^3) e^{i\omega_o T_o} \\ & + \frac{1}{2}(H_1 \bar{A} + 3H_2 A\bar{A}^2) e^{i(\Omega - \omega_o)T_o} + (\frac{H_2}{2}\bar{A}^3) e^{i(\Omega - 3\omega_o)T_o} + NST. + c.c. \end{aligned} \qquad \dots (7)$$

where $NST.$ denotes the terms does not produce secular terms and $c.c.$ denotes the complex conjugate. Any particular solution of Eq.(7) contains secular terms, which are generated by the first term on the right-hand side of Eq.(7). Moreover, it may contains small-divisor terms depending on the resonance condition. Eq.(7) contains two conditions to obtain periodic solutions $(\Omega \cong n\omega_o)$; $n = 2,4$. i.e their exist sub-harmonic periodic solutions of even order $(\frac{1}{2}, \frac{1}{4})$.

3 Sub-harmonic solution of order $\frac{1}{2}(\Omega \cong 2\omega_o)$

In this section, we study subharmonic solution of order $\frac{1}{2}$. i.e periodic solutions with period equal two multiple of the period of the excitation term i.e ($\Omega \cong 2\omega_o$). Introducing the detuning parameter σ in Eq.(7) to convert the small divisor term into secular term.

$$i.e \quad \Omega = 2\omega_o + \varepsilon\sigma \qquad \dots (8)$$

and write

$$(\Omega - \omega_o)T_o = \omega_o T_o + \varepsilon\sigma T_o = \omega_o T_o + \sigma T_1, T_1 = \varepsilon T_o \qquad \dots (9)$$

Using (9), the small-divisor term arising from $exp\, i(\Omega - 2\omega_o)$ in Eq.(7) can be transformed into a secular term. Then, eliminating the secular terms yield

$$2i\omega_o A' + 2i\omega_o \zeta A + F_1 A + 3F_3 A^2 \bar{A} + 10F_5 A^3 \bar{A}^2 + 35F_7 A^4 \bar{A}^3 + \frac{1}{2}(H_1 \bar{A} + 3H_2 A\bar{A}^2)e^{i(\sigma T_1)} = 0 \quad \ldots (10)$$

One can take

$$A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)} \quad \ldots (11)$$

where a and β are real. Inserting Eq.(11) into Eq.(10) and separate real and imaginary parts, we obtain

$$a' = -a\zeta - \frac{1}{4\omega_o}(H_1 a + \frac{3H_2}{4}a^3)\sin\gamma \quad \ldots (12)$$

$$a\gamma' = a\sigma - \frac{1}{\omega_o}(F_1 a + \frac{3}{4}F_3 a^3 + \frac{5}{8}F_5 a^5 + \frac{35}{64}F_7 a^7) - \frac{1}{4\omega_o}(H_1 a + \frac{3H_2}{4}a^3)\cos\gamma \quad \ldots (13)$$

where Equations (12) and (13) represent the modulation equations in the amplitude and the phase.

$$\gamma = \sigma T_1 - 2\beta \quad \ldots (14)$$

It is obvious that, Eqs.(12) and (13) have a trivial solution which corresponds to the trivial steady state solution. Non-trivial steady state solution corresponds to the non-trivial fixed points(Equilibrium points) of Eqs. (12) and (13). That is, they satisfy $a' = \gamma' = 0$, and are given by

$$\frac{1}{4\omega_o}(H_1 a_o + \frac{3H_2}{4}a_o^3)\sin\gamma_o = -a_o\zeta \quad \ldots (15)$$

$$\frac{1}{4\omega_o}(H_1 a_o + \frac{3H_2}{4}a_o^3)\cos\gamma_o = a_o\sigma - \frac{1}{\omega_o}(F_1 a_o + \frac{3}{4}F_3 a_o^3 + \frac{5}{8}F_5 a_o^5 + \frac{35}{64}F_7 a_o^7) \quad \ldots (16)$$

where a_o, γ_o correspond to steady state solutions. Eliminating $\sin\gamma_o$ and $\cos\gamma_o$ from Eqs.(15) and (16) yields the *frequency-response* equation

$$(a_o\sigma - \frac{1}{\omega_o}(F_1 a_o + \frac{3}{4}F_3 a_o^3 + \frac{5}{8}F_5 a_o^5 + \frac{35}{64}F_7 a_o^7))^2 + (a_o\zeta)^2 - \frac{1}{16\omega_o^2}(H_1 a_o + \frac{3}{4}H_2 a_o^3)^2 = 0 \quad \ldots (17)$$

From(17), we get:

$$\sigma = \frac{64F_1 + 48a_o^2 F_3 + 40a_o^4 F_5 + 35a_o^6 F_7 \pm 8\sqrt{16H_1^2 + 24a_o^2 H_1 H_2 + 9a_o^4 H_2^2 - 256\zeta^2\omega_0^2}}{64} \quad \ldots (18)$$

The first-order uniform expansion of the solution (first approximation) of Eq.(1) is given by

$$u = a\cos(\frac{1}{2}\Omega t - \frac{1}{2}\gamma) + O(\varepsilon) \quad \ldots (19)$$

The analysis of the stability of the trivial solutions is equivalent to the analysis of the linear solutions of equation(10) by neglecting the non-linear terms we get

$$2i\omega_o A' + 2i\omega_o\zeta A + F_1 A + \frac{1}{2}H_1 \bar{A} e^{i(\sigma T_1)} = 0 \quad \ldots (20)$$

To determine the stability of the trivial steady state solution, it is convenient to rewrite A in the form

$$A = (B(T_1) + ib(T_1))e^{i\frac{1}{2}\sigma(T_1)} \quad \ldots (21)$$

where B and b are separates real and imaginary parts, we get

$$b' + \zeta b + \Gamma_1 B = 0 \quad \ldots (22)$$

$$B' + \zeta B - \Gamma_2 b = 0 \quad \ldots (23)$$

Eqs.(22)and(23) admit solution of the form $(B,b) \propto (B,b)e^{\theta_o T_1}$, where (B,b) are constants. The eigenvalues of the coefficient matrix of Eqs.(22) and (23) are

$$\theta_o = -\zeta \pm \sqrt{\Gamma_1 \Gamma_2}. \qquad \text{… (24)}$$

where $\Gamma_1 = (\frac{1}{2}\sigma - \frac{1}{2\omega_o}F_1 - \frac{1}{4\omega_o}H_1)$ and $\Gamma_2 = (\frac{1}{2}\sigma - \frac{1}{2\omega_o}F_1 + \frac{1}{4\omega_o}H_1)$. Then, the trivial solution is stable if the real parts of both eigenvalues are less than or equal zero. To determine the stability of the non-trivial steady state solutions given by Eqs.(15) and (16). Let

$$a = a_0 + a_1(T_1) \quad \& \quad \gamma = \gamma_0 + \gamma_1(T_1) \qquad \text{… (25)}$$

where a_0 and γ_0 correspond to a non-trivial steady state solutions and a_1 and γ_1 are perturbations which are assumed to be small compared with a_0 and γ_0. Substituting (25) into equations (12) and (13) and linearizing the resulting equations, we obtain

$$a_1' = -a_1(\zeta + (m_1 + 3m_2 a_o^2)\sin\gamma_o) - \gamma_1((m_1 + 3m_2 a_o^3)\cos\gamma_o) \qquad \text{… (26)}$$

$$\begin{aligned}\gamma_1' = \ & a_1[\frac{\sigma}{a_o} - \frac{1}{a_o\omega_o}(F_1 + \frac{9}{4}F_3 a_o^2 + \frac{25}{16}F_5 a_o^4 + \frac{245}{128}F_7 a_o^6) - \frac{1}{a_o}(m_1 + 3m_2 a_o^2)\cos\gamma_o] \\ & - \gamma_1[(m_1 + m_2 a_o^2)\sin\gamma_o]\end{aligned} \qquad \text{… (27)}$$

where $m_1 = \frac{H_1}{4\omega_o}$ and $m_2 = \frac{3H_2}{16\omega_o}$. Equations (26) and (27) admit solution of the form $(a_1, \gamma_1) \propto (d_1, d_2)e^{\theta T_1}$ where (d_1, d_2) are constants. Provided that,

$$\begin{aligned}\theta = \ & -\frac{4\zeta H_1}{4H_1 + 3a^2 H_2} \pm \frac{1}{32\sqrt{2}}([(\frac{1}{(4H_1 + 3a^2H_2)^2\omega_o^2}(z_1F_1F_5H_1 + z_2F_3F_5 + z_3F_1F_7H_1^2 + z_4F_3F_7 \\ & + z_5F_5F_7 + z_{10}F_3^2 + z_8F_5^2 + z_7F_1^2H_2 + z_6F_1F_3 + z_9F_7^2 + z_{11}\sigma F_3\omega_o + z_{12}\sigma F_5H_1\omega_o + z_{13}\sigma F_7\omega_o \\ & + z_{14}\sigma F_1H_2\omega_o + z_{15}\zeta^2\omega_o^2 + z_{16}\sigma^2H_2\omega_o^2))]^{\frac{1}{2}})\end{aligned} \qquad \text{… (28)}$$

where

$z_1 = -40960a^4H_1 - 30720a^6H_2, \quad z_2 = -46080a^6H_1^2 - 46080a^8H_1H_2 - 8640a^{10}H_2^2,$

$z_3 = -53760a^6H_1^2 - 53760a^8H_1H_2 - 10080a^{10}H_2^2, z_4 = -53760a^8H_1^2 - 60480a^{10}H_1H_2 - 15120a^{12}H_2^2,$

$z_5 = -56000a^{10}H_1^2 - 67200a^{12}H_1H_2 - 18900a^{14}H_2^2, z_6 = 13824a^6H_2^2 - 24576a^2H_1^2,$

$z_7 = 24576a^2H_1 + 18432a^4H_2, z_8 = -7200a^{12}H_2^2 - 28800a^{10}H_1H_2 - 25600a^8H_1^2,$

$z_9 = -11025a^{16}H_2^2 - 29400a^{12}H_1^2 - 36750a^{14}H_1H_2, z_{10} = -18432a^4H_1^2 - 13824a^6H_1H_2,$

$z_{11} = 24576a^2H_{1\,o}^2 - 13824a^6H_2^2, z_{12} = 40960a^4H_{1_o} + 30720a^6H_2,$

$z_{13} = 53760a^6H_1^2 + 53760a^8H_1H_2 + 10080a^{10}H_2^2, z_{14} = -49152a^2H_1H_2 - 36864a^4H_2^2,$

$z_{15} = 32768H_1^2 + 98304a^2H_1H_2 + 73728a^4H_2^2 and z_{16} = 24576a^2H_1 + 18432a^4H_2$

and $z_{16} = 24576a^2H_1 + 18432a^4H_2$.

The solution is stable if and only if the real part of each of the eigenvalues of the coefficient of the matrix are less than or equal to zero.

4 Sub-harmonic solution of order $\frac{1}{4}(\Omega \cong 4\omega_o)$

In this section, we study subharmonic solution of order $\frac{1}{4}$ i.e periodic solutions with period equal four multiple of the period of the excitation term i.e ($\Omega \cong 4\omega_o$). Introducing the detuning parameter σ_1 in Eq.(7) to convert the small divisor term into secular term.

$$i.e \quad \Omega = 4\omega_o + \varepsilon\sigma_1 \quad \dots (29)$$

, and write

$$(\Omega - 3\omega_o)T_o = \omega_o T_o + \varepsilon\sigma_1 T_o = \omega_o T_o + \sigma_1 T_1, \quad T_1 = \varepsilon T_o \quad \dots (30)$$

Eliminating the secular terms form the Eq.(9) yields

$$2i\omega_o A' + 2i\omega_o \zeta A + F_1 A + 3F_3 A^2 \overline{A} + \frac{1}{2}(H_2 \overline{A}^3)e^{i(\sigma_1 T_1)} = 0 \quad \dots (31)$$

Using the polar form $A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)}$ like in the previous section into the Eq.(31) and separating real and imaginary parts, we obtain the following modulation equations: In Eq.(31), where a and β are real, separate real and imaginary parts, and obtain

$$a' = -a\zeta - \frac{1}{16\omega_o}(H_2 a^3)\sin\phi \quad \dots (32)$$

$$a\phi' = a\sigma_1 - \frac{1}{\omega_o}(F_1 a - \frac{3}{4}F_3 a^3 - \frac{5}{8}F_5 a^5 - \frac{35}{64}F_7 a^7) - \frac{1}{16\omega_o}(H_2 a^3)\cos\phi \quad \dots (33)$$

where Eqs.(32)and (33) represent the modulations in the amplitude and the phase, $\phi = \sigma_1 T_1 - 2\beta$ for steady state solution, $a' = \phi' = 0$, in Eqs.(32)and(33) we obtain

$$a_o\zeta = -\frac{1}{16\omega_o}(H_2 a_o^3)\sin\phi_o \quad \dots (34)$$

$$a_o\sigma_1 - \frac{1}{\omega_o}(F_1 a_o + \frac{3}{4}F_3 a_o^3 + \frac{5}{8}F_5 a_o^5 + \frac{35}{64}F_7 a_o^7) = -\frac{1}{16\omega_o}H_2 a_o^3 \cos\phi_o \quad \dots (35)$$

where a_o and γ_o correspond to steady state solutions. Eliminating $\sin\phi_o$ and $\cos\phi_o$ from Eqs.(34) and (35) yields the *frequency-response* equation

$$(a_o\sigma_1 - \frac{1}{\omega_o}(F_1 a_o + \frac{3}{4}F_3 a_o^3 + \frac{5}{8}F_5 a_o^5 + \frac{35}{64}F_7 a_o^7))^2 + (a_o\zeta)^2 - \frac{1}{16\omega_o^2}(H_2 a_o^3)^2 = 0 \quad \dots (36)$$

From (36), we get

$$\sigma_1 = \frac{64F_1 + 48a_o^2 F_3 + 40a_o^4 F_5 + 35a_o^6 F_7 \pm \sqrt{\frac{1}{16\omega_{o^2}}a_o^4 H_2^2 - 4\zeta^2}}{\omega_0} \quad \dots (37)$$

The first-order uniform expansion of the solution (first approximation) of Eq.(1) is given by

$$u = acos(\frac{1}{4}\Omega t - \frac{1}{4}\gamma) + O(\varepsilon) \quad \dots (38)$$

Now, the analysis of the stability of the trivial solutions is equivalent to the analysis of the linear solutions of equation (10) by neglecting the non -linear terms we get

$$2i\omega_o A' + 2i\omega_o\zeta A + F_1 A = 0 \quad \dots (39)$$

To solve Eq.(39) and lets $A = (B(T_1) + ib(T_1))e^{i\sigma_1(T_1)}$, where B and b are real and imaginary parts and get

$$b' + \zeta b + (\frac{1}{2}\sigma_1 - \frac{1}{2\omega_o}F_1)B = 0 \quad \dots (40)$$

$$B' + \zeta B - (\frac{1}{2}\sigma_1 - \frac{1}{2\omega_o}F_1)b = 0 \quad \dots (41)$$

Eqs.(40)and(41) admit solution of the form $(B,b) \propto (b_1,b_2)e^{\theta_o T_1}$, where (b_1,b_2) are constants. The eigenvalues of the coefficient matrix of Eqs.(40)and(41) are

$$\theta_o = -\zeta \pm i\Gamma_1 \quad \dots (42)$$

where ($\Gamma_1 = \frac{1}{2}\sigma_1 - \frac{1}{2\omega_o}F_1$) Then, the trivial solution is stable if the real parts of both eigenvalues are less than or equal to zero.

To determine the stability of the non-trivial steady state solutions given by Eqs.(32) and (33), we use Eqs.(34) and (35). Let

$$a = a_0 + a_1(T_1) \qquad \& \qquad \phi = \phi_0 + \phi_1(T_1) \quad \dots (43)$$

where a_o and ϕ_o are given by Eqs.(34) and (35). Inserting Eq.(43)into Eqs.(32) and (33) and using Eqs.(34) and (35) and keeping only the linear terms in a_1 and ϕ_1, we get

$$a_1' = a_1(\zeta + 3m_1 a_o^2 \sin\phi_o) - \phi_1(m_1 a_o^3 \cos\phi_o) \quad \dots (44)$$

$$a_o\phi_1' = a_1(\sigma_1 - \frac{1}{\omega_o}(F_1 + \frac{9}{4}F_3 a_o^2 + \frac{25}{8}F_5 a_o^4 + \frac{245}{64}F_7 a_o^6) - 3m_1 a_o^2 \cos\phi_o) + \phi_1(m_1 a_o^3 \sin\phi_o) \quad \dots (45)$$

where $m_1 = \frac{H_2}{16\omega_o}$. Equations (44) and (45) admit solution of the form $(a_1,\phi_1) \propto (c_1,c_2)e^{\theta T_1}$ where (c_1,c_2) are constants. Provided that,

$$\begin{aligned}\theta = -\zeta \pm \frac{1}{32}([(\frac{1}{\omega_o^2}(2048F_1^2 + 1536a^2F_1F_3 - 960a^6F_3F_5 - 800a^8F_5^2 \\ -1120a^6F_1F_7 - 1680a^8F_3F_7 - 2100a^{10}F_5F_7 \\ -1225a^{12}F_7^2 - 2048\sigma_1F_1\omega_o - 768a^2\sigma_1F_3\omega_0 \\ +560a^6\sigma_1F_7\omega_o + 9216\zeta^2\omega_o^2 + 512\sigma_1^2\omega_o^2))]^{\frac{1}{2}})\end{aligned} \quad \dots (46)$$

Consequently, a solution is stable if and only if the real parts of both eigenvalues (46) are less than or equal to zero.

5 Numerical results and discussions

This section presents numerical results for sub-harmonic solutions of order $\frac{1}{2}$ (one-to-two) and $\frac{1}{4}$ (one-to-fourth) by solving the frequency response equations (18) and (37) and stability conditions (24), (28), (42) and (46) for different values of the parameter in the equations. The numerical results are plotted in a group of Figures (1-24), which represent the variation of the amplitude a with the detuning parameter (σ and σ_1) for given values of the other parameters in which solid (dashed) curves, represent stable (unstable) solutions. Figures (1-12) represent the frequency-response curves of the subharmonic solutions of order $\frac{1}{2}$. In (Fig.1), we note that, the response amplitude has two branches which are bent to the right so that the upper branch has stable for large values and there exist a jump phenomena(a saddle node bifurcation) and the lower branch has unstable solution. For increasing the coefficient of the linear term F_1 respectively, we observe that the two branches are shift to the right so that the regions of definition, stability and multivalued are decreased, (Fig.3). When F_1 is decreased with negative values respectively, we note that two branches are shift to the left and the saddle node bifurcations are exist at the points $(\sigma = -7.5, \sigma = -5.5, \sigma = -3.5)$. The zones of definition, stability and multi-valued are increased, (Fig.4). As the coefficient of the cubic term F_3 takes the values (0.9, 2, 3), we note that two branches are shift downwards and have decreased magnitudes respectively and there exist a jump phenomena (a saddle node bifurcations) are exist at the points $(\sigma = -1.5, \sigma = -0.9, \sigma = -0.4)$. The regions of definition, stability and multivalued are decreased, Fig.5. For decreasing F_3 respectively, we observe that the upper branch shifts to the left and move to the upper and the lower branch shifts upwards. The two branches have increased magnitudes and there exist a saddle node bifurcations at the points $(\sigma = -4.4, \sigma = -3.5, \sigma = -2.7)$. The regions of definition, stability and multivalued are increased, (Fig.6). For the coefficient of the quantic term F_5 increasing and decreasing respectively, we get the same variation as in Figures(5, 6) and Figs.(7,8). As the coefficient of the seventh nonlinear term F_7 takes the values (3, 5, 7), we note that the two branches are shift downward and have decreased magnitudes respectively. The saddle node bifurcation are exist at the points($(\sigma = -1.49, \sigma = -1.11, \sigma = -0.89)$),(Fig.9). (Fig.19), represent the variation of F_7 for the same values with magnitudes values (-3, -5, -7) when $F_7 = -3$, we deserve that the two branches are bent to the left. Also, the stability for the trivial solution and by examination of the eigenvalues Eq.(4.13) for nontrivial solution for the second case. For further decreasing F_7, of the two branches are shift downwards and have decreased magnitudes. When the coefficient of linear parametric H_1 is increased respectively, we note that the upper branch shift to the left and has increased magnitude respectively and there exist a saddle node bifurcations at the points ($\sigma = -4.3, \sigma = -5.4, \sigma = -6.8$). The lower branch shifts to the right and has decreased magnitudes respectively, (Fig.11). For increasing the coefficient of nonlinear parametric excitation H_2, we get the same variation as in (Fig.7) so that the saddle node bifurcation exist at the points $(\sigma = -3.13, \sigma = -4.3, \sigma = -5.6)$, (Fig.12).

Figures (13-24) represent the frequency response curves for sub-harmonic solutions of order one-fourth. In (Fig.13), the response amplitude has multivalued curves which bent to the right. The multivalued curve consists of two branches so that the upper branch has unstable solution for small values and stable solution for higher values and there exist a saddle node bifurcation at ($\sigma_1 = -3.6$). The lower branch has unstable solution. The minimum point exist at($\sigma_1 = 0$). When F_1 takes the values(0.01, 1 and 3), we note that the multivalued curve shift to the right and move to downwards and has decreased magnitudes so that the minimum point shifts to the right respectively. The saddle node bifurcation are exist at the points ($\sigma_1 = -3.6, \sigma_1 = -1.7, \sigma_1 = -2.3$). The regions of multivalued, stability and definition are decreased, (Fig.15). As F_1 is decreased with negative values respectively, we note that the multivalued curve shift to the left move upward respectively and the minimum point move to the left respectively. The saddle node bifurcation exist at the points ($\sigma_1 = -11.7, \sigma_1 = -5.7, \sigma_1 = -3.7$). The regions of multivalued, stability and definition are increased, (Fig.16). For decreasing F_3 respectively, we note that the multivalued curve shifts to the left and move upward respectively so that the minimum value shifts nearest to ($\sigma_1 = 0$). The region of definition, multivalued and stability are increased, (Fig.18). when F_3 take values (0.01, 1, 3), we observe that the multivalued curve contracted respectively so that the upper branch has decreased magnitudes and the lower branch has increased magnitudes. The region of definition, multivalued and stability are decreased the upper branch, (Fig.19). For increasing and decreasing F_5, respectively, we get the same variation as in Figures(19, 20) so that the minimum point does not affect and exist at the same point ($\sigma = 0$), Figs.(20, 21). For increasing F_7, we get the same variation as in (Fig.18) and (Fig.22), when F_7 take negative value ($i.e. F_7 = -0.1$), we observe that the multivalued curves is bent to the left so that the saddle node bifurcation exist at the point ($\sigma_1 = 5.7$). As F_7 is decreased further,

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the multivalued curve shift downwards so that it has the same magnitude in the region and after this interval it has decreased magnitudes. The regions of definition, stability and multivalued are decreased, (Fig.22), H_2 is increased, we note that the multivalued curve is contracted and inclosed inside the main multivalued curves so that the minimum point move upwards and takes increased value respectively, (Fig.23). For increasing the damping factor ζ, we note the multivalued curve is contracting and given semi-oval so that the upper branch has stable solutions and has decreased magnitudes in small interval and after this interval it has the same magnitudes. The lower branch has increased unstable magnitudes. As $\zeta = 3$, the semi-oval is contracted so that the upper branch has stable solutions and the lower branch has unstable increased magnitudes. The zones of multivalued, definition and stability are decreased, (Fig.24).

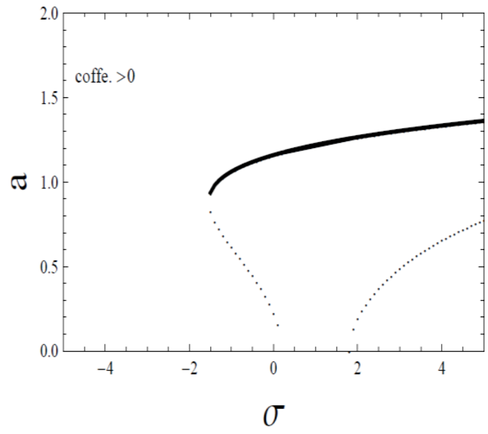

Fig.1

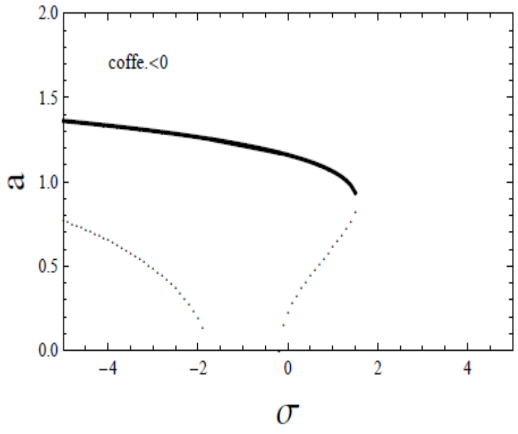

Fig.2

The frequency response curves of the sub-harmonic solution of order $\frac{1}{2}$ *for the parameters* $\omega_o = 1, \zeta = 0.3, F_1 = 1, F_3 = 0.9, F_5 = 0.1, F_7 = 3, H_1 = 2, H_2 = 10$

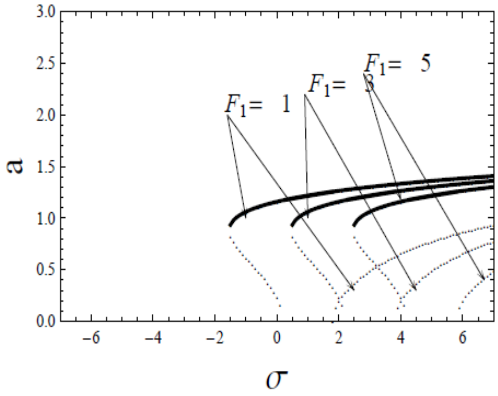

Fig. 3

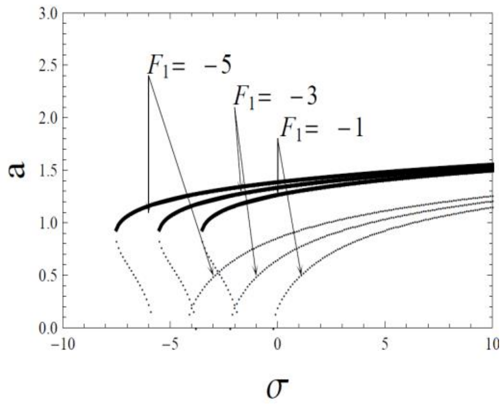

Fig. 4

Variation of the amplitude of the response with the detuning parameter for increasing and decreing F_1

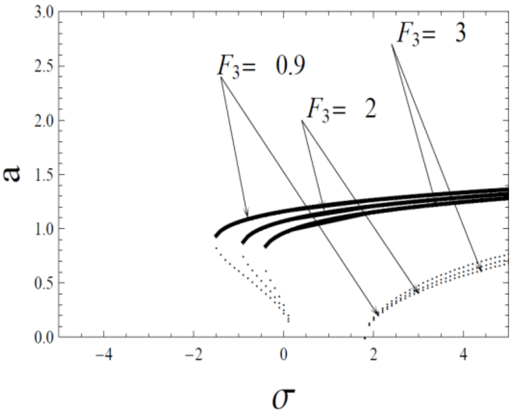

Fig. 5

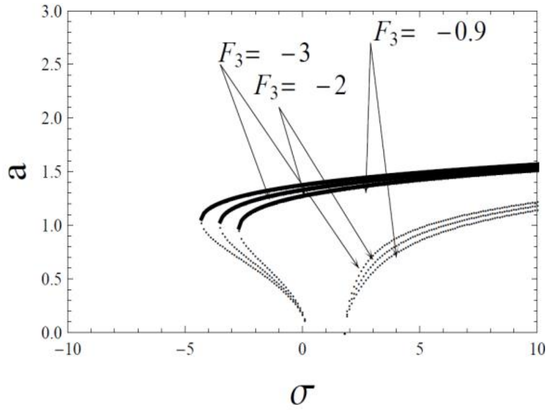

Fig. 6

Variation of the amplitude of the response with the detuning parameter for for increasing and decreasing F_3

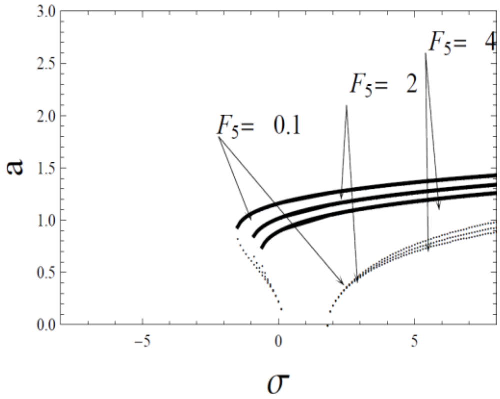

Fig.7

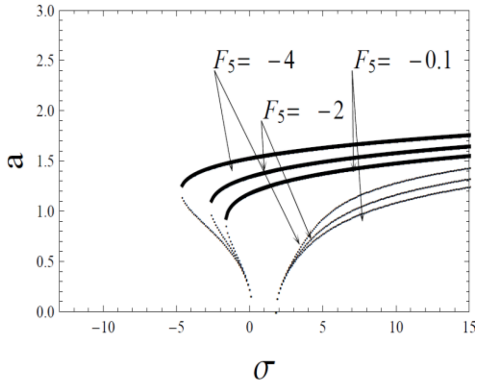

Fig. 8

Variation of the amplitude of the response with the detuning parameter for for increasing and decreasing F_5

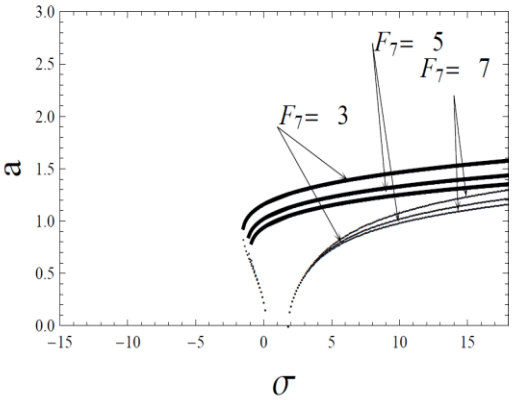

Fig. 9

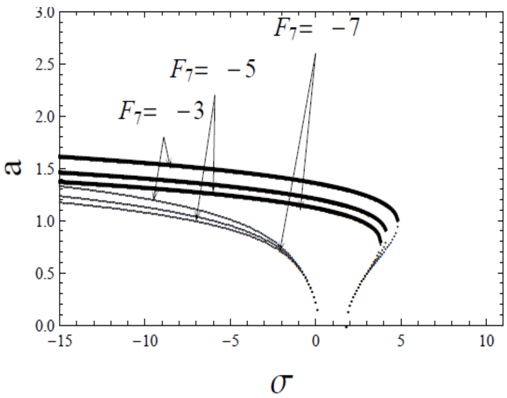

Fig.10

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_7

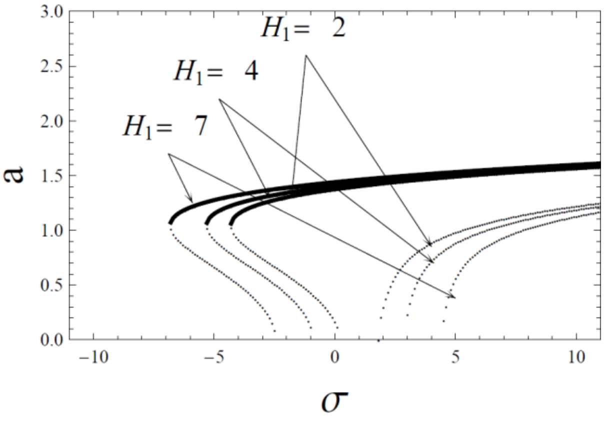

Fig. 11
Variation of the amplitude of the response with the detuning parameter for increasing H_1

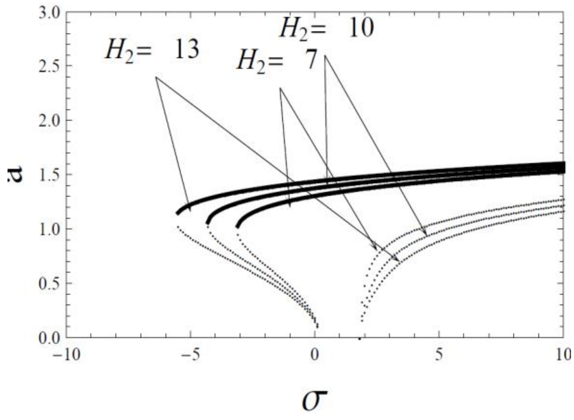

Fig. 12
Variation of the amplitude of the response with the detuning parameter for increasing and decreasing H_2

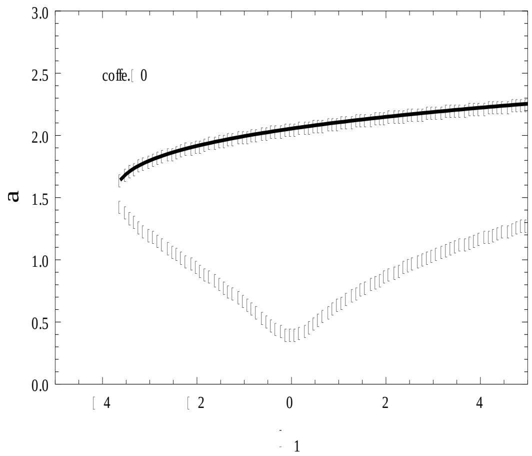

Fig. 13

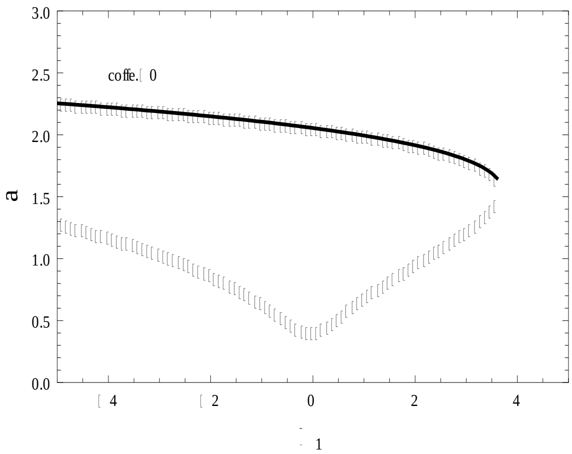

Fig. 14

The frequency response curves of the sub-harmonic solution of order $\frac{1}{4}$ *for the parameters* $\omega_o = 1, \zeta = 0.1, F_1 = 0.01, F_3 = 0.01, F_5 = 0.1, F_7 = 0.1, H_2 = 10$

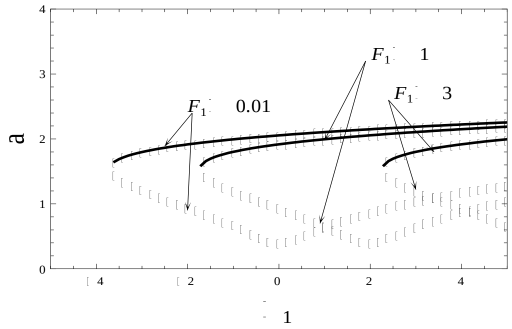

Fig.15

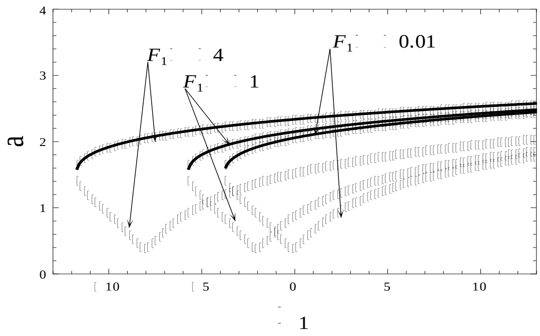

Fig.16

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_1

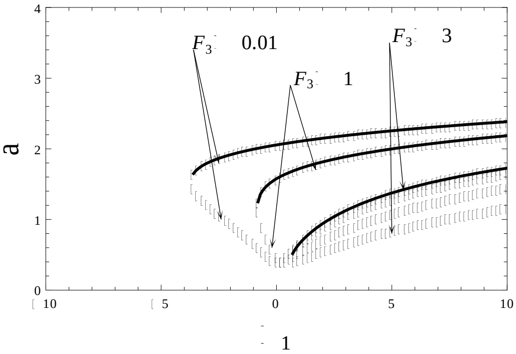

Fig.17

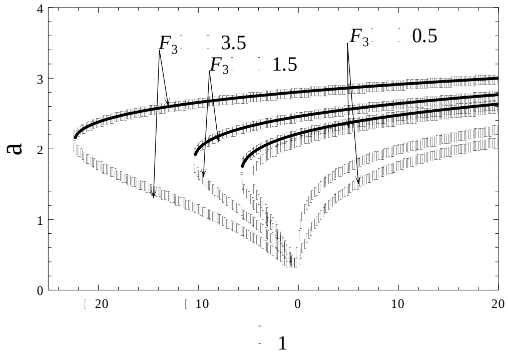

Fig.18

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_3

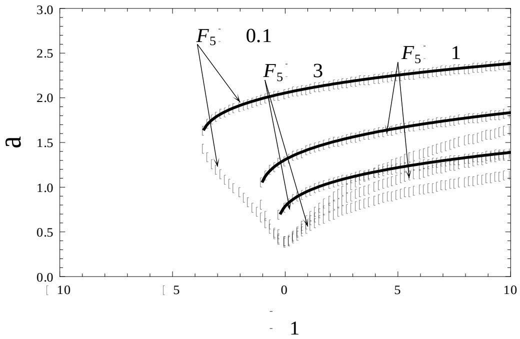

Fig.19

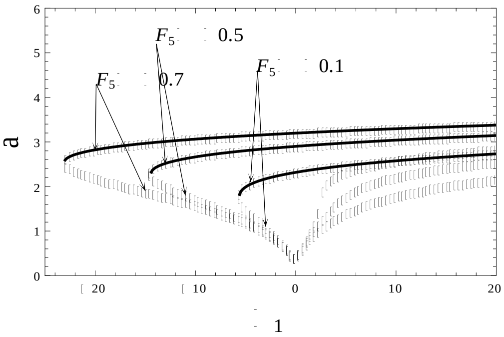

Fig.20

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_5

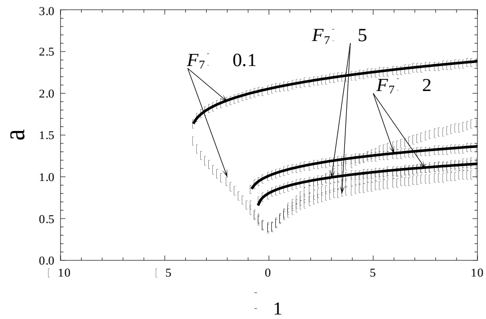

Fig. 21

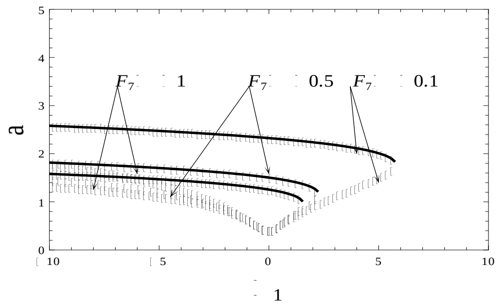

Fig. 22

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_7

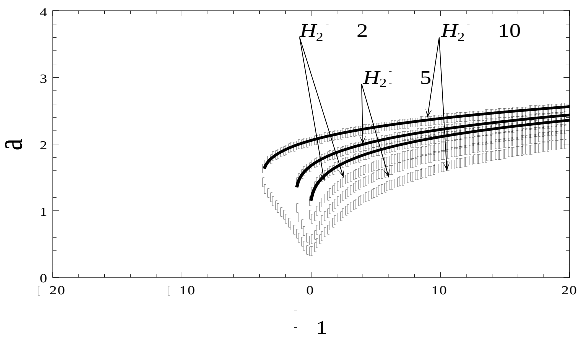

Fig. 23
Variation of the amplitude of the response with the with the detuning parameter for decreasing H_2

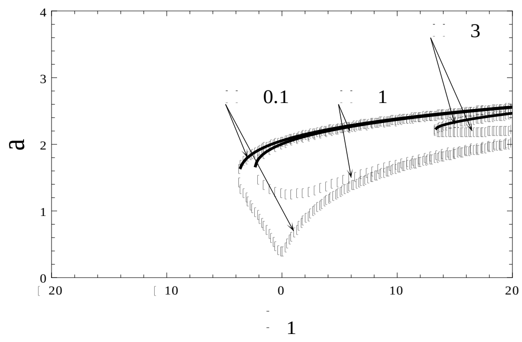

Fig. 24
Variation of the amplitude of the response detuning parameter for increasing ζ

6 Conclusion

Sub-harmonic solutions of even order $(\frac{1}{2},\frac{1}{4})$ to a weakly second order differential equation governed the motion of a micro-electro mechanical system(MEMS) (Bandpass Filter) are investigated analytically. Applying the perturbation method(Multiple scales method) approximate expressions up to $O(\varepsilon)$ are obtained. For each types of periodic solutions the modulation equations in the amplitude and the phases, the steady-state solutions, the frequency response equations and the stability conditions are determined. Numerical calculations and the results are plotted in group of Figures, stable(unstable) solutions solid(dashed) curves and represent stable(unstable) solutions. Generally their exist two solutions higher stable and lower unstable and their exist jump phenomena. Bending of the curves in the Right direction of σ axis.

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